Regular Grammars and Closure Properties of Regular Languages

CS154

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Outline

- Regular Grammars
- Closure Properties of Regular Languages

Grammars

- We now consider a different way to look at the regular languages based on grammars.
- A **grammar** is defined as a 4-tuple G=(V, T, S, P) where V is a finite set of **variables**, T is a finite set of **terminal** symbols, $S \in V$ is called the start variable, and P is a finite set of **productions** of the form v --> w where v is in $(V \cup T)^+$ and w is in $(V \cup T)^*$.
- For example, let G=({<sentence>,<noun>,<verb>}, {dog, cat, walks, eats}, <sentence>, P) where P is

  <sentence> --> <noun> <verb>

  <sentence> --> <noun> <verb> <noun>

  <noun> --> dog | cat /* we are using | to abbreviate two line <noun> --> dog and <noun> --> cat */

  <verb> --> walks | eats

  <noun> <verb> --> <sentence>
- Beginning with the start variable, a grammar can **yield** or **generate** a string over the alphabet of terminals via a finite sequence of substitutions:

  <sentence> ==> <noun> <verb> ==> dog <verb> ==> dog walks

  We write <sentence> ==>* dog walks to indicate from the string <sentence> we can get dog walks via a finite sequence of substitutions.
- We write L(G) for the set of strings generated by a grammar.

Regular Grammars

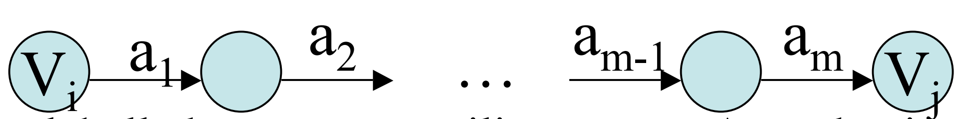
- A grammar G=(V, T, S, P) is called **right-linear** if all its productions are of the form A --> xB or A--> x for some A, B in V and x in T*.
- A grammar is called **left-linear** if all its productions are of the form A --> Bx or A-->x for some A, B in V and x in T*.
- A grammar is called **regular** if it is either left or right linear.
- For example, G=({S}, {a,b}, S, P) where P contains S--> abS | λ is right linear. It generates the strings in $(ab)^*$.
- For example, G=({S, A}, {a,b}, S, P) where P contains S--> Sab |A, A--> Aba | ba is left linear. It generates the strings in $(ba)^+(ab)^*$.
- The set of rules S-->A, A--> aB| λ, A--> Ab are all linear (so could belong to a **linear grammar**). The second rule is right linear, and the third is left linear, so these rules together could not belong to either a right linear or left linear grammar.

Equivalence with Regular Languages

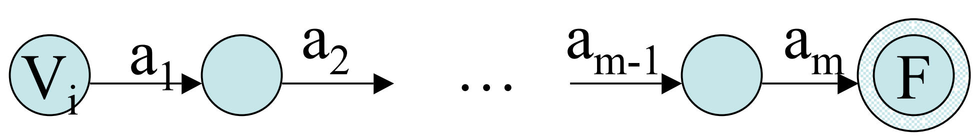
- Need to show every language generated by a regular grammar is regular and vice-versa.
- In class we will only look at right linear grammars, but a similar argument can be made for left linear grammars. To begin:

Theorem. Let G=(V, T, S, P) be a right linear grammar. Then L(G) is a regular language.

Proof. Let V={V_0, .., V_n}. Assume S=V_0. The alphabet of our NFA will be the set of terminals. The set of states of our NFA will consist of V_0, .., V_n together with some auxiliary states and the state f which will be the unique accepting state. The start state will be V_0. The transition function δ will be based on the productions of G. A production V_i --> $a_1 \ldots a_m$ V_j will map to the sequence of states and transitions:



where the unlabelled states are auxiliary states. A production of the form V_i --> $a_1 \ldots a_m$ is mapped to a set of transitions:



Given this description of the NFA, one can observe that V_0 ==>* w if and only if $\delta^*(V_0, w) = f$ and so if and only if w is accepted by the NFA.

Regular implies Regular Grammar

Theorem. If L is a regular language, then it is generated by some regular grammar.

Proof. Let M = $(Q, \Sigma, \partial, q_0, F)$ be a DFA for L. Assume $Q=\{q_0, \ldots, q_n\}$ and $\Sigma=\{a_0, \ldots, a_m\}$. Let $G=(V,\Sigma, S, P)$ be the grammar with $V=\{q_0, \ldots, q_n\}$ and $S= q_0$ and where for each transition $\partial(q_i, a_j) = q_k$ we have the production $q_i \dashrightarrow a_j q_k$ and if q_k is in F we also have the production q_k-->λ. It is not hard to see that w is accepted by M iff it is generated by this right linear grammar.

Closure Properties

- Last day we argued that the regular languages are closed under union, concatenation and *.
- Today, we will look at some further closure properties.
- To begin…

Theorem. The regular languages are closed under complement.

Proof. A regular language L is accepted by some DFA $M=(Q, \Sigma, \partial, q_0, F)$. Let $M'=(Q, \Sigma, \partial, q_0, Q-F)$. This machine will accept precisely those strings in Σ^* which are not accepted by M. i.e., $\overline{L}$.

Direct Product Construction for DFAs

Theorem. If A_1 and A_2 two regular languages, so is their intersection $A_1 \cap A_2$.

Proof: Let $M_1=(Q_1, \Sigma, \delta_1, q_1, F_1)$ and $M_2=(Q_2, \Sigma, \delta_2, q_2, F_2)$ be the DFAs recognizing A_1 and A_2. We would like make a new DFA, M, which simultaneously simulates both M_1 and M_2 and accepts a string w if both M_1 and M_2 accepts. To simulate both machines at the same time we use a so-called cartesian product construction. Let $Q = Q_1 \times Q_2$. M's alphabet is Σ like that of M_1 and M_2. Define $\delta((q, q'), a) = (\delta_1(q,a), \delta_2(q',a))$. Let the start state be (q_1, q_2). Finally, let $F = (F_1 \times F_2)$.

Corollary. If A_1 and A_2 two regular languages, so is $A_1 - A_2$.

Proof. Notice $A_1 - A_2 = A_1 \cap \overline{A}_2$.

Closure under Reversals

Theorem. If L is regular then so is L^R.= $\{w^R \mid w$ is in L$\}$. Here w^R is w written backwards.

Proof. Let N=$(Q, \Sigma, \partial, s, F)$ be an NFA for L. Recall from our proof that L(N) can be generated by a regular expression, that we can assume N has only one accept state f. Let N´ be the NFA obtained from N by making f the start state, s the only accept state, and for each transition $\partial(q, a) = q'$ having instead the transition $\partial(q', a) = q$. This machine will recognize a string iff the reverse was in N.