1. Given 2 regular languages $L_1$ and $L_2$, we want to prove $L_1 \cap L_2$ is also a regular language.

By definition of regularity, there exists DFA $M_1$ and $M_2$ such that $M_1$ recognizes $L_1$ and $M_2$ recognizes $L_2$.

Let $M_1 = (Q_1, \Sigma, \delta_1, q_{10}, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_{20}, F_2)$.

Let's construct a new machine $M = (Q, \Sigma, \delta, q_0, F)$ such that $Q = Q_1 \times Q_2$, $\Sigma = \Sigma$, $q_0 = (q_{10}, q_{20})$, $F = \{(q, p) | q \in F_1 \land p \in F_2\}$.

Let's now confirm that our machine recognizes $L_1 \cap L_2$.

We want $(L_1 \cap L_2)(M) = (L_1 \cap L_2)$.

Notice $\delta^*(q, q', w) = (\delta_1^*(q, w), \delta_2^*(q', w))$.

WeL_1$ iff $\delta_1^*(q, w) \in F_1$ and $\delta_2^*(q', w) \in F_2$.

WeL_2$ iff $\delta_2^*(q', w) \in F_2$.

So if weL_1 \cap L_2, then $\delta_1^*(q, w) \in F_1$ and $\delta_2^*(q', w) \in F_2$.

And thus $(\delta_1^*(q, w), \delta_2^*(q', w)) \in F$.

But $(\delta_1^*(q, w), \delta_2^*(q', w)) = \delta^*(q, q', w)$.

So $M$ accepts all weL_1 \cap L_2.

Now if $w \notin L_1 \cap L_2$, then we $\notin L_1 \lor w \notin L_2$, so either $\delta_1^*(q, w) \notin F_1$ or $\delta_2^*(q', w) \notin F_2$.

Thus $\delta^*(q, q', w) \notin F$.

And since $(\delta_1^*(q, w), \delta_2^*(q', w)) = \delta^*(q, q', w)$, weL_1 \cap L_2 $\Rightarrow$ $M$ does not accept $w$.

M recognizes $L_1 \cap L_2$ and it is a DFA therefore $L_1 \cap L_2$ is regular. QED

2. $L = \{w \mid w \text{ begins with } 010^3\}$

$L' = \{w \mid w \text{ ends with } 101^3\}$

NFA for $L$
NFA for $L^n$

![NFA Diagram]

<table>
<thead>
<tr>
<th>CurrentStates</th>
<th>NextStates</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_0$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$q_1$</td>
<td>$q_0$, $q_2$</td>
</tr>
<tr>
<td>$q_2$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$q_3$</td>
<td>$q_2$</td>
</tr>
</tbody>
</table>

We can now stop checking $\varepsilon$ transitions for $q_0$, $q_1$, $q_2$, and $q_3$ since they all yield $\emptyset$.

CurrentStates does not contain an accept state after processing $w = 1011010$, thus $w$ is not a member of the language.
4. Let our NFA for \( L(U) \) be \( N = (Q, \Sigma, \delta, q_0, F) \).

We will construct a DFA, \( D = (Q', \Sigma, \delta', q_0', F') \).

Let \( Q' = \mathcal{P}(Q) \).

Let \( q''_0 = \delta(q_0, \epsilon) \).

Let \( q''_e = \delta(q, \epsilon) \).

Let \( F' = \{ q' | \epsilon \rightarrow q' \text{ contains an accept state of } N \} \).

Many states in the DFA are unreachable, however.

Since \( N \) only has one node with multiple outgoing edges,
and for every other state, \( q, \delta(q) = \emptyset \),
and removing that node would result in a graph that is
not connected, we only have to consider the states
\( \{ q_0, q_1, q_2, q_3 \} \times \{ q_0, q_3, q_1, q_2 \} \cup \delta(q) \).

We start at \( q_0'' \) and follow all paths of 0's and 1's.
Now we want to minimize the DFA.

$(q_0^1, q_0^2)$ is distinguishable

$(q_0^1, q_0^2, q_0^3)$ is distinguishable

$(q_0^1, q_1, q_2, q_3)$ is distinguishable

$(q_0^1, q_3, q_0^2, q_3)$ is distinguishable

$(q_0^1, q_3, q_0^2)$ is distinguishable.

$(q_0^1, q_3, q_0^2, q_3)$ is distinguishable because $(q_0^1, q_3, q_0^1, q_3)$ is distinguishable.

End up with $[q_0^1], [q_0^2], [q_2, q_3], [q_3, q_0^1], [q_3, q_3].$

Applying $6(q_0, a) = [q_3]$. results in the DFA:

(minimized DFA)

5. Make a GNFA from our DFA.

Removing $[q_0^1]$ by comparing legal path through $[q_0^1]$

between every start node $u$ and end node $v$ and using it with the length 1 path from $u$ to $v$.

I will evaluate $X|U^0$ as $X$ and $X|O^0$ as $0$ and $2a$ as a
Now we will remove the $[\text{Eq}_1, \text{Eq}_3]$ node using the same process.

Now we will remove $[\text{Eq}_1, \text{Eq}_3]$ using the same process.
Now $[\text{Eq. 1.3}]$

6. Let $G = (V, S, F)$ where

$V = \{[q_1], [q_2], [q_3], [q_4], [q_5]\}$

$p = \{[q_1] \rightarrow O([q_2]), [q_3] \rightarrow O([q_2]), [q_4] \rightarrow O([q_2]), [q_5] \rightarrow O([q_2]), [q_1] \rightarrow [q_1], [q_2] \rightarrow [q_2], [q_3] \rightarrow [q_3], [q_4] \rightarrow [q_4], [q_5] \rightarrow [q_5]\}$

$S = [q_0]$  

7. Let $h(0) = \emptyset$, $h(1) = \epsilon$, $h(2) = \epsilon$

Regex for $L$: $(1(\cup 2))^*0(1(\cup 2))0(1(\cup 2))^*0(1(\cup 2))^*$

Construct $L'$ by applying $h$ on regex for $L$

$(\epsilon \cup \text{zero} \cup \text{zero} \cup \text{zero})^*$

$(\epsilon \cup \text{zero} \cup \text{zero} \cup \text{zero})^*$

$(\epsilon \cup \text{zero} \cup \text{zero} \cup \text{zero})^*$
\( G = (V, E, P, S) \) where
8. Proof by induction

Base step: Level $0$ has

$$L_{\leq 0} = \{w | w = "v" \}$$

This language is regular because

$$L(C(wjw...w)) = L_{\leq 0}$$

Inductive step:

Assume $L_{\leq n}$ is regular.

Let $R$ be the regular expression that describes $L_{\leq n}$.

Then the regular expression for

$$L_{n+1}$$

is

$$\left( \left( wjw...w \right)^* \right)^n$$

So the regular expression describing

$$L_{\leq n+1}$$

is

$$\left( \left( wjw...w \right)^* \right)^n \cup R$$

So $L_{\leq n+1}$ is regular.

We have proven $L_{\leq 0}$ is regular and

$L_{\leq n}$ being regular implies $L_{\leq n+1}$

is regular.

Therefore $L_{\leq n}$ is regular for all $n \in \mathbb{N}$.

QED
10. Assume \( L_n \subseteq \Sigma^* \) is regular, then it has a pumping length \( p \).

Consider \( w = v^i a = p, w \in L_n \). Using the pumping lemma splitting \( w = xyz \)

\(|xyz| \leq p\) so \( x = \ldots \) and \( y = \ldots \) s.t. \( i > 0 \), and \( z = \ldots \) as \( a = a = a = a \),

\( xy^2z \) must be in \( L_n \) by the pumping lemma,

but \( xy^i z = \ldots \) which is equivalent to \( p + i + j \) because \( j > 0 \)

so \( xy^2z \) is not in the language. Thus we have a contradiction and \( L_n \)

is not regular.

QED