Exercises:

1. Let $A=\{1,3,7,8,9\}$ and $B=\{2,3,4,8\}$.
   Write out fully the elements in each of the following sets
   (a) $A \cap B$
   (b) $B - A$
   (c) $S(A)$
   (d) 2^B
   (e) $A \times B$.
   Give the cardinality of each set.

   (a) $A \cap B = \{3,8\}$
      $|A \cap B| = 2$
   (b) $B - A = \{2,4\}$
      $|B-A| = 2$
   (c) $S(A) = \{1,3,7,8,9\} \cup \{\{1,3,7,8,9\}\} = \{1,3,7,8,9,\{1,3,7,8,9\}\}$
      $|S(A)| = 6$
   (d) $2^B = \{\varnothing, \{2\}, \{3\}, \{4\}, \{8\}, \{2,3\}, \{2,4\}, \{2,8\}, \{3,4\}, \{3,8\}, \{4,8\}, \{2,3,4\}, \{2,3,8\}, \{2,4,8\}, \{3,4,8\}, \{2,3,4,8\}\}$
      $|2^B| = 16$
   (e) $A \times B = \{(1,2), (1,3), (1,4), (1,8)$
            $(3,2), (3,3), (3,4), (3,8)$
            $(7,2), (7,3), (7,4), (7,8)$
            $(8,2), (8,3), (8,4), (8,8)$
            $(9,2), (9,3), (9,4), (9,8)\}$
      $|A \times B| = 20$

2. All possible partitions of set {1,2,3,4,5}

1|2|3|4|5
12|3|4|5
13|2|4|5
14|2|4|5
15|2|3|4
23|1|4|5
24|1|3|5
25|1|4|5
34|1|2|5
35|1|2|4
45|1|2|3
123|4|5
124|3|5
125|3|4
134|2|5
135|2|4
145|2|3
234|1|5
235|1|4
245|1|3
345|1|2
123|45
124|35
125|34
134|25
135|24
145|23
234|15
235|14
245|13
345|12
1234|5
1235|4
1245|3

1345|2
2345|1
12|34|5
12|35|4
12|45|3
13|24|5
13|25|4
13|45|2
14|23|5
14|25|3
14|35|2
15|23|4
15|24|3
15|34|2
23|45|1
24|35|1
25|34|1
12345

3. Construct using ∧, ∨, ¬ gates a boolean function $\{0,1\}^4 \to \{0,1\}$ which returns 1 if and only if all but one of its inputs have the same value.

P	Q	R	S	Z	
0	0	0	0	0	
0	0	0	1	1	¬P∧¬Q∧¬R∧S
0	0	1	0	1	¬P∧¬Q∧R∧¬S
0	0	1	1	0	
0	1	0	0	1	¬P∧Q∧¬R∧¬S
0	1	0	1	0	
0	1	1	0	0	
0	1	1	1	1	¬P∧Q∧R∧S
1	0	0	0	1	P∧¬Q∧¬R∧¬S
1	0	0	1	0	
1	0	1	0	0	
1	0	1	1	1	P∧¬Q∧R∧S
1	1	0	0	0	
1	1	0	1	1	P∧Q∧¬R∧S
1	1	1	0	1	P∧Q∧R∧¬S
1	1	1	1	0	

P'Q'R'S + P'Q'RS' + P'QR'S' + P'QRS + PQ'R'S' + PQ'RS + PQR'S + PQRS'

P'(Q'R'S + Q'RS' + QR'S'+ QRS) + P(Q'R'S' + Q'RS + QR'S + QRS')

P'(Q'(R'S + RS') + Q(R'S'+ RS)) + P(Q'(R'S' + RS) + Q(R'S + RS'))

in math notation:
¬P ^ (¬Q ^ (¬R^S v R^¬S) v Q ^ (¬R^¬S v R^S)) v P ^ (¬Q ^ (¬R^¬S v R^S) v Q ^ (¬R^S v R^¬S))

4. Prove by induction that, $\sum_{i=0}^{n} 3i^2+3i = (n+1)^3-(n+1)$. Show carefully that this sum is $\Theta(n^3)$.

Base step:

for n = 1: $(3(0)^2 + 3(0)) + (3(1)^2 + 3(1)) = 0 + 3 + 3 = 6$
$(1+1)^3 - (1+1) = 8 - 2 = 6$
For n=1, the equality proves true.

Inductive hypothesis:

Assume the equality proves true for n=k.

$$\sum_{i=0}^{k} 3i^2+3i \quad = \quad (k+1)^3-(k+1)$$

Inductive proof for n = k+1:

$$(k+1)^3-(k+1)+[3(k+1)^2+3(k+1)]=((k+1)+1)^3-((k+1)+1)$$

$$(k+1)^3-(k+1)+[3(k^2+2k+1)+3k+3]=(k+2)^3-(k+2)$$

$$(k^2+2k+1)(k+1)-(k+1)+[3k^2+6k+3+3k+3]=(k^2+4k+4)(k+2)-(k+2)$$

$$(k^3+2k^2+k+k^2+2k+1)-(k+1)+[3k^2+9k+6]=(k^3+4k^2+4k+2k^2+8k+8)-(k+2)$$

$$k^3+3k^2+3k+1-k-1+3k^2+9k+6=k^3+6k^2+12k+8-k-2$$

$$k^3+6k^2+11k+6=k^3+6k^2+12k+8-k-2$$

$$k^3+6k^2+11k+6=k^3+6k^2+11k+6$$

The equality holds true for n = k+1.

Conclusion:

By induction, $\sum_{i=0}^{n} 3i^2+3i \;=\; (n+1)^3-(n+1)$.

Proof that $(n+1)^3-(n+1)$ is $\Theta(n^3)$:

Expanded, $(n+1)^3-(n+1)$ is $n^3+6n^2+11k+6$

f(n) = $n^3+6n^2+11k+6$

g(n) = n^3

Prove Big O: for some positive c_1 and n_0, prove that

$f(n)\le c_1 g(n)$, where $n\ge n_0$

$$n^3+6n^2+11k+6\le c_1 n^3$$

$n^3+6n^2+11k+6\le 5n^3$ where $n\ge 10$

$$10^3+6(10^2)+11(10)+6\le 5(10^3)$$

$1716\le 5000$ for $c_1=5, n_0=10$

Prove Big Omega: for some positive c_2 and n_0, prove that

$f(n)\ge c_2 g(n)$, where $n\ge n_0$

$n^3+6n^2+11k+6 \geq c_2 n^3$

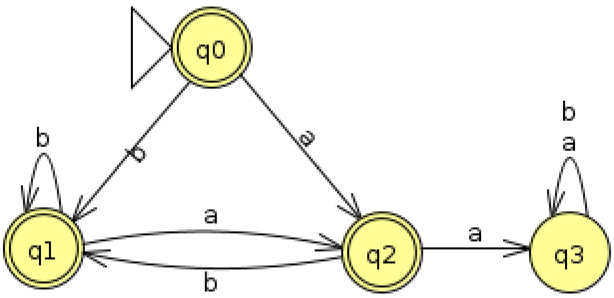
$n^3+6n^2+11k+6 \geq 1n^3$ where $n \geq 10$

$10^3+6(10^2)+11(10)+6 \geq 1(10^3)$

$1716 \geq 1000$ for $c_2=1, n_0=10$

Thus, $(n+1)^3-(n+1)$ is $\Theta(n^3)$ for positive constants $c_1=5, c_2=1, n_0=10$.

5. Diagram:



5-tuple notation for doubleA finite automata:

1. Q = {q0, q1, q2, q3}
2. Σ = {a, b}
3. δ is described as

	a	b
q0	q2	q1
q1	q2	q1
q2	q3	q1
q3	q3	q3

4. q0 is the start state
5. F = {q0, q1, q2}

6. The pigeonhole principle states that if there are n holes and n+1 or more pigeons, then at least one hole contains two pigeons. For this automata, it has 3 non-initial states, or "holes to fill". The pigeons in this case are the tokens of the string given to the automata. For this automata, the longest fixed length before repetition of a state would be 3+1, or 4 tokens long. However, if we consider only the strings which the machine accepts (and not rejects), then the holes to fill become 2 (q1 and q1) and the longest string length becomes 2+1 = 3.

McCulloch and Pitt Summary

In this paper, McCulloch and Pitt attempt to model neurons and neural events with propositional logic – every reaction of a neuron is thought of as a simple proposition. They present their assumptions about the neurons within the net – most importantly that the activity of each neuron is all-or-none, and others like the net structure is not affected by time and no significant delay is present besides synaptic delay. They model a net of neurons N by designating each one as c_1, c_2, and so forth, similar to defining the states of a finite automata. They designate actions of neurons to be N_1, N_2, and so on, representing whether or not a neuron will fire – a predicate where if a neuron is firing, is true. They then proceed to present theorems about the behavior of neuron networks using combinations of propositional logic and sentences built using Language II as defined by Carnap. They present the concept of realizability, which determines if a net will be able to compute a given predicate or combination of them. They define TPEs, or temporal propositional expressions, which are predicates with a free variable which is time, and show that all TPEs are realizable by nets without circles (cycles), and there are indefinitely many topologically different nets realizing the same TPEs.

Questions:

1. The system of logic and base theory used in the paper is not set theory. Where do the authors say it comes from?

The system used is called Language II, by Carnap, along with additional notations from Russel and Whitehead’s Principia. The base theory is propositional (boolean) logic.

2. Where in the paper (page/paragraph) is something like a finite automata defined?

On page 4, paragraph 2, the authors define a net N with something like states represented by c1, c2, etc. In the previous and next paragraphs, something like a language for that net is defined, and general concepts of a net providing a solution for sentences, similar to how an automata can accept or reject strings from a language. If a net accepts a sentence, that sentence is “realizable”. Each neuron has a predicate associated with it, representing whether it will fire or not, and essentially leading to the next neuron/”state” within the net; these predicates can be considered as a transition function (I think). I’m not entirely sure about if/how TPEs from the paper fit in.

3. In what other subject areas do you suppose this paper appears as one of the foundational papers?

It is likely foundational for neural networks in machine learning, and possibly computational neuroscience.