Types of Splines

CS116B
Chris Pollett
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Outline

- Spline Representations
- Cubic-Spline Interpolation

Spline Representations

- Last day we said how to represent a spline using its parametric cubic (x(u), y(u)) where u is in [0,1] , the endpoint values (x(0), y(0)), (x (1), y(1)) and derivatives at the endpoints (x'(0), y'(0)) and (x'(1), y'(1)). Here x(u) looks like: $a_x*u^3 + b_x*u^2 + c_x*u + d_x$. and y(u) is similar.
- Today we'll look at different ways to represent this same equations

Other Representations

- Note could write x(u) as:

  $[u^3\ u^2\ u\ 1][a_x\ \ b_x\ \ c_x\ d_x]^t = \mathbf{U}\cdot\mathbf{C}$ (the little t is for transpose)
- One can also write C as $\mathbf{M_{spline}} \cdot \mathbf{M_{geom}}$ where $\mathbf{M_{spline}}$ is a the boundary values on the spline and $\mathbf{M_{geom}}$ contains a 4x4 matrix based on the coordinates of the control points.
- Thus, x(u) is sometimes written as $\mathbf{U}\cdot \mathbf{M_{spline}} \cdot\mathbf{M_{geom}}$ which is called the *basis matrix*.
- Finally, sometimes see splines represented in terms of coordinates of control points as $x(u)=\Sigma^3_{k=0}\ g_k \cdot BF_k\{u\}$ where BF_k are called blending functions.

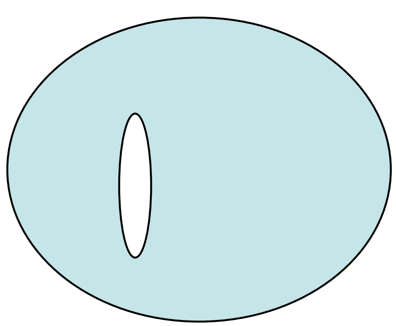
Spline Surfaces

- The splines discussed so far were 2D. Can also specify surfaces.
- To do this need to specify sets of orthogonal spline curves using a mesh of control points.
- Might have an equation like:

$$P(u,v) := \Sigma_{i,j}\ p_{i,j} BF_i(u) BF_j(v)$$

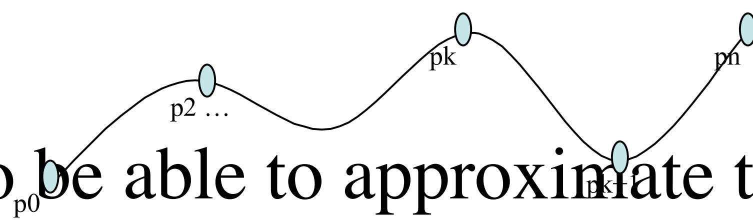
Trimming Spline Surfaces

- Sometimes in CAD applications it is useful to be able to trim out sections from a spline surface.
- Some graphics packages have facilities to generate **trimming curves** to support this



Cubic-Spline Interpolation Methods

- Rather than using general splines of arbitrary degree, cubic splines are often used to design objects because they are reasonably flexible and can be computed and stored efficiently.
- Consider the curve



- One wants to be able to approximate this curve with cubic splines.

More on Cubic Spline Interpolation

- Between each pair of control points p_k, p_{k+1} we will try to find a best approximating cubic spline. (x(u), y(u), z(u)) where x(u), y(u), z(u) are cubics in u.
- To do this we need to set enough boundary conditions at the endpoints of a segment so this spline is uniquely determined.
- Next few slides discuss different ways to do this

Natural Cubic Splines

- In these kind of spline, if have n+1 control points then we specify n cubic splines.
- We specify the values of the spline, its first and second derivative, at each of its endpoints.
- We require adjacent splines to have matching values at the endpoints.
- To complete the description usual set the first and second derivative of p_0 and p_n to be 0.

Hermite Interpolation

- Natural splines are hard to locally update
- For Hermite splines we specify the value of the tangent at each control point rather than say that this value must be equal among adjacent curves.
- For example, if want to specify curve P between two points p_k, $p_{k+1,}$ would use equations:

  $P(0)= p_k$

  $P(1) = p_{k+1}$

  $P'(0) = Dp_k$ Here Dp_k is some fixed value like 4.

  $P'(1) = Dp_{k+1}$

More on Hermite Interpolation

Nnow $\mathbf{P}(u) = \mathbf{a}u^3 + \mathbf{b}u^2 + \mathbf{c}u + \mathbf{d}$ for $0<=u<=1$

- That is $\mathbf{P}(u) = [u^3\ u^2\ u\ 1][\mathbf{a\ b\ c\ d}]^t$
- Taking the derivative we have $\mathbf{P'(u) :=}$
  $[3u^2\ 2u\ 1\ 0]\ [\mathbf{a\ b\ c\ d}]^t$
- Substituting values 0 and 1 for u in the above gives us the matrix equation:

$$\begin{bmatrix} \mathbf{p_k} \\ \mathbf{p_{k+1}} \\ \mathbf{p_k} \\ \mathbf{p_{k+1}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{bmatrix}$$

Call the inverse matrix $\mathbf{M_{H.}}$ Can solve now for **a b c d** by taking $\mathbf{M_{H.}}$

Yet More on Hermite Interpolation

- So $\mathbf{M_H}$ is:

$$\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- $\mathbf{P}(u) = [u^3\ u^2\ u\ 1]\ \mathbf{M_H}\ [\mathbf{p_k}\ \mathbf{p_{k+1}}\ \mathbf{Dp_k}\ \mathbf{Dp_{k+1}}]^t$
- From which we can calculate the blending functions:

$$\mathbf{P}(u) = \mathbf{p_k}(2u^3 - 3u^2 + 1) + \mathbf{p_{k+1}}(-2u^3 + 3u^2) + \mathbf{Dp_k}(u^3 - 2u^2 + u) + \mathbf{Dp_{k+1}}(u^3 - u^2)$$

$$\mathbf{P}(u) = \mathbf{p_k}H_0(u) + \mathbf{p_{k+1}}H_1(u) + \mathbf{Dp_k}H_2(u) + \mathbf{Dp_{k+1}}H_3(u)$$