# 3D transformations

CS116A
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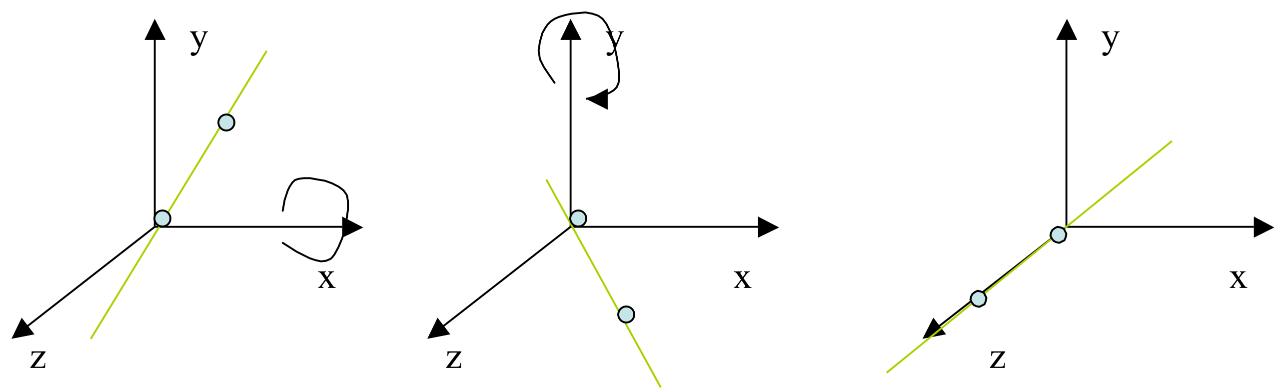
# Outline

- More on 3D Rotations
- Fun with complex numbers
- Quaternions
- Composite 3D transformations
- Other 3D transformations

# General 3D rotations

- Recall from last day we said to do a general rotation we:
  - Translate the object so that the rotation axis passes through the coordinate origin
  - Rotate the object so that the axis of rotation coincides with the z-coordinate axes. To do this:
    - rotate around x-axis until point is in xz-plane
    - rotate around y-axis until point is aligned with z-axis
  - Perform the specified rotation
  - Perform the inverses of the first two steps.

# Pictures of Hard steps



# How to do the hard steps

- Let $u=(a,b,c)$ be the unit vector in the direction of the point we want to rotate into the xz plane.
- The angle $\alpha$ we rotate is the same angle as to rotate $u' = (0, b, c)$ into the xz as component along x direction fixed by x-axis rotation
- Only need to figure out $\cos \alpha$ and $\sin \alpha$ to do the rotation. Let $i=(1,0,0)$, $j=(0,1,0)$, $k=(0,0,1)$
- $\cos \alpha = u' . k/|u'||k| = c/d$ where $d=(b^2+c^2)^{1/2}$
- Using the cross product, $u' \times k$, one can show $\sin \alpha$ is $b/d$.

# More hard step

- So the rotation matrix looks like:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- u’’=(a,0,d) is the unit vector in the xz plane in the direction of our vector after the above rotation. We want to rotate this vector to the z-axis.
- Need sin and cos of angle β needed to do this.
- Again, dot and cross product.

# Yet more hard step

- $\cos\ \beta = u'' \cdot k / |u''||k| = d$.
- Can show using cross product, $u'' \times k$ that $\sin\ \beta = -a$
- So the last rotation we do is:

$$\begin{bmatrix} d & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The complete sequence of rotations is:

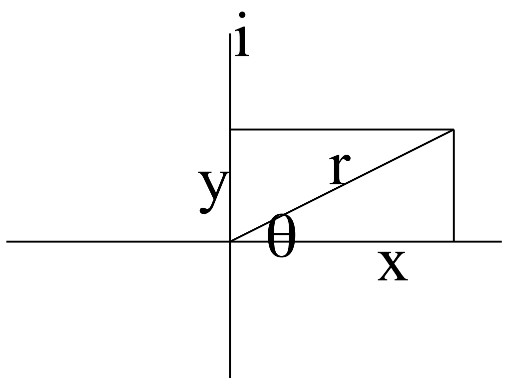
$$R(\theta) = T^{-1}R^{-1}_x(\alpha)R^{-1}_y(\beta)R_z(\theta)R_y(\beta)R_x(\alpha)T$$

# Fun with complex numbers

- We are going to consider some properties of complex numbers as a warm-up to doing rotations using quaternions.

# Complex Numbers

- Recall a complex number $z=x+iy$ can be used to represent a point in the plane $(x,y)$.




- This can alternative be written in polar coordinates as $re^{i\theta}$
- Here r=[(x+iy)(x-iy)]^{1/2}.We call \bar z = x -iy the conjugate of z.

# Inverses

- Given a complex number $z= re^{i\theta}$, its inverse can be computed as $z^{-1}=(1/r)e^{-i\theta}$.
- So geometrically the map $z \to z^{-1}$ maps points in the upper half plane outside of the unit circle to point in the lower half plane interior to the unit circle.
- Points on the unit circle itself are reflected horizontally.

# Powers

- Consider the map $z \to z^k$.
- So $re^{i\theta} \to r^k e^{ik\theta}$. The effect is both a stretching radially in the plane coupled with a rotation of the point by k times its initial angle.

# Remark

- Just adding one complex number to another allows us to do translations. So it seems plausible could represent plane transformations as addition and multiplication of complex numbers. Does something like this exist for 3-space?
- We care about the answer because multiplying by complex numbers is cheaper than by matrices, so it should be for the 3D analog. Also, 3D analog useful for solving some problems like gimbal lock.

# Quaternions

- Are numbers of form $q=a+ bi + cj +dk$.
- Can be used to represent points in homogeneous coordinates by saying $q=q'$ if there is a real h such $hq=ha+hbi+ hcj +hdk = q'$
- Multiplication of two quaternions, $q.q'$, can be figured out using the rule $i^2=-1$, $j^2=-1$, $k^2=-1$ and that $ij = i \times j$ (cross product) where i and j are the unit vectors in in x and y directions respectively.(Similarly, for other possible combinations).
- Multiplication is not commutative but is associative.
- The length $||q||$ of a quaternion q is $(a^2+b^2+c^2+d^2)^{1/2} = [(a+ bi + cj +dk)(a- bi - cj -dk)]^{1/2}$. Here $q^*=a-bi-cj-dk$ is the conjugute of q

# Inverse

- $q^{-1} = q^{*}/\|q\|^{2}$

# Representing 3DRotations

- For 3D vector v, write q=(s,v) to denote q = s +v_1i + v_2j + v_3k.
- Suppose we want to rotate about a 3D unit vector u by an angle θ.
- Let s= cos(θ/2) and let v=u * sin (θ/2) and let q= (s,v). Note this is a unit quaternion.
- Then to rotate a point P = ai+bj+ck compute: q^{-1}Pq.

# Composite 3D transformations

- To do several 3D transformations in one go we just multiply the corresponding matrices together. For example, try to work out: M=R_z(θ)T(x,y,z)

# 3D Reflections

- Reflections in 3D are about a plane rather than a line as in 2D.
- For example, to reflect a point through the xy-plane could use the matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# 3D Shear

- Like 2D case except now can also shear in z-direction:

$$
\begin{bmatrix}
1 & 0 & sh\_zx & -sh\_zx*z0 \\
0 & 1 & sh\_zy & -sh\_zy*z0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$