Theorem 1. The step-multigraph \( S(n; a_1, a_2, \ldots, a_k) \) with \( n \) odd is edge-graceful.

We see immediately from Theorem 1 that the conjecture above is true for step-multigraphs \( S(n; a_1, a_2, \ldots, a_k) \) with \( n \) odd and decomposable into \( k \) Hamiltonian cycles.

2. Preliminaries.

Lo [13] shows that if \( G \) is a simple \((p, q)\)-graph with \( p = |V(G)| \) and \( q = |E(G)| \) and \( G \) is edge-graceful, then \( p \) divides \( q^2 + q - \lfloor p(p - 1) \rfloor / 2 \). This result can be extended to multigraphs.

Theorem 2. If \( G \) is a \((p, q)\)-multigraph with \( p = |V(G)| \) and \( q = |E(G)| \) and \( G \) is edge-graceful, then \( p \) divides \( q^2 + q - \lfloor p(p - 1) \rfloor / 2 \).

The first author, Li-Min Lee and G. Murty [6] showed that any simple graph \( G \) with \( p \equiv 2 \pmod{4} \) is not edge-graceful. We also have the similar result for multigraphs.

Theorem 3. A multigraph \( G \) with \( p \equiv 2 \pmod{4} \) is not edge-graceful.

For other results on edge-graceful graphs, we refer the readers to [5 - 13].

It is easy to see that the step-multigraph \( S(n; a_1, a_2, \ldots, a_k) \) with \( n \) odd satisfies the necessary condition as stated in Theorem 2.

For an edge \((u, v)\) in the step-multigraph \( S(n; a_1, a_2, \ldots, a_k) \) we define the vertex difference of this edge to be \((v - u) \pmod{n}\), which must be an element of the set \(\{a_1, a_2, \ldots, a_k\}\). For a given vertex \( u \), the set of vertex differences for all edges incident with \( u \) is a union of two identical copies of \(\{a_1, a_2, \ldots, a_k\}\).

We partition the edges of \( S(n; a_1, a_2, \ldots, a_k) \) into \( k \) classes such that the edges in the \( i \)-th class, \( A_i \), has the same vertex difference \( a_i \), where \( i = 1, 2, \ldots, k \). That is,

\[ A_i = \{(u, v) : u, v \in V(G), u \neq v, v - u \equiv a_i \pmod{n}\} \]

Note that each class has exactly \( n \) edges. Note also that if we have \( a_i = a_j \) for \( i \neq j \), we still end up with two distinct classes \( A_i \) and \( A_j \), although the edges in the two classes are the same. It is not difficult to see that the edges in each of these classes form either a Hamiltonian cycle, a union of disjoint cycles, or a union of disjoint edges.

Since the vertex labels are calculated by using modulo \( n \) arithmetic, we may replace the set of edge labels \(\{1, 2, \ldots, nk\}\) by the set \(\{1 \mod{n}, 2 \mod{n}, \ldots, nk \mod{n}\}\). That is, we can use \( k \) copies of \( B = \{0, 1, \ldots, n - 1\} \) as the edge labels.

3. Labeling techniques.

In this section, we introduce two labeling techniques, which we use to label the edges of \( S(n; a_1, a_2, \ldots, a_k) \). The edges in a class \( A_i \) will be labeled by one of the techniques.