

for  $G$ , we have  $|v(0) - v(1)| \leq 1$  for  $(G, H)$ . Similarly,  $e(0) = e(1)$  for  $H$  and  $|e(0) - e(1)| \leq 1$  for  $G$ . Consider the augmenting edges between  $G_i$  and  $H_i$ ,  $1 \leq i \leq \ell$ . Each vertex in  $G_i$  is incident with an even number of such edges. Since  $v(0) = v(1)$  for  $H_i$ , half of these edges are labelled 0 and half labelled 1, regardless of the label on the vertex in  $G_i$ . Hence,  $|e(0) - e(1)| \leq 1$  for  $(G, H)$ . The converse can be proved in the same way. ■

Note that apart from being cordial, there are no other restrictions placed on  $G$ . The subsets  $G_1, G_2, \dots, G_\ell$  are completely arbitrary. They do not have to be distinct or disjoint, and their union does not have to include every vertex of  $G$ .

In our applications of Theorem 1, we use the following graphs in the role of  $H$ .  $T_{2k}$  is the trivially cordial graph consisting of  $2k$  isolated vertices. We take  $H_1 = T_{2k}$ .  $T_{4k}$  is the trivially cordial graph consisting of  $4k$  isolated vertices. Here, each of  $H_1$  and  $H_2$  consists of  $k$  vertices labelled 0 and  $k$  labelled 1.  $M_{4k}$  is cordial by (7). Here,  $H_1$  consists of all vertices on one side of the matching arranged so that exactly half of them are labelled 0.  $H_2$  consists of all vertices on the other side of the matching. Other choices of  $H$  will also be encountered.

**Theorem 2.** All generalized fans  $F_{m,n}$  (see Figure 1(g)) are cordial.

Proof: Take  $H = T_{2k} = H_1$ . If  $m = 2k$ , take  $G = P_n = G_1$ . If  $m = 2k + 1$ , take  $G = F_n$  and  $G_1 = P_n$ . Then  $(G, H) = F_{m,n}$ . Since  $P_n$  is cordial by (5) and  $F_n$  is cordial by (8),  $F_{m,n}$  is cordial by Theorem 1. ■

**Theorem 3.** A bundle  $B_n$  (see Figure 1(f)) is cordial if and only if  $n \not\equiv 2 \pmod{4}$ .

Proof: If  $n \equiv 2 \pmod{4}$ , then  $B_n$  is Eulerian. Since its number of edges is  $3n \equiv 2 \pmod{4}$ ,  $B_n$  is not cordial according to (2).

Now take  $H = M_{4k}$  with  $H_1$  and  $H_2$  defined as before. If  $n = 4k + i$ ,  $i = 1, 3, 4$ , take  $G = B_i$ .  $G_1$  consists of one of the vertices of  $B_i$  not of degree 2, and  $G_2$  consists of the other vertex not of degree 2. Then  $(G, H) = B_n$ . It is easy to verify that  $B_1, B_3$  and  $B_4$  are cordial. By Theorem 1, so is  $B_n$ . ■

**Theorem 4.** A generalized bundle  $B_{m,n}$  (see Figure 1(i)) is cordial if and only if  $m \equiv 0 \pmod{2}$  or  $n \not\equiv 2 \pmod{4}$ .

Proof: Suppose  $n \equiv 2 \pmod{4}$ . Take  $H = T_{4k}$  with  $H_1$  and  $H_2$  defined as before. If  $m = 2k$ , take  $G = M_n$ .  $G_1$  consists of vertices on one side of the matching and  $G_2$  consists of the vertices on the other side. If  $m = 2k + 1$ , take  $G = B_n$ . Let  $G_1$  and  $G_2$  be as in the case  $m = 2k$ . Note that the two vertices of  $B_n$  not of degree 2 are not in  $G_1$  or  $G_2$ . Then  $(G, H) = B_{m,n}$ . Since  $M_n$  is cordial by (7) and  $B_n$  is cordial by Theorem 3,  $B_{m,n}$  is cordial by Theorem 1.

Suppose  $n = 4k + 2$  and  $m \equiv 0 \pmod{2}$ . Take  $H = B_{m-2,4k}$ , using the cordial labelling obtained above. Take  $G = B_{2,2}$ , which is shown to be cordial in

Figure 2. The same figure defines  $G_i$  and  $H_i$ ,  $1 \leq i \leq 4$ , in the construction of  $(G, H) = B_{m,n}$ . By Theorem 1, it is a cordial graph.

Finally, if  $n \equiv 2 \pmod{4}$  and  $m \equiv 1 \pmod{2}$ , then  $B_{m,n}$  is Eulerian. Since its number of edges is  $2m + n \equiv 2 \pmod{4}$ ,  $B_{m,n}$  is not cordial according to (2). ■

**Theorem 5.** If  $m \equiv 1 \pmod{2}$ , a generalized wheel  $W_{m,n}$  (see Figure 1(h)) is cordial if and only if  $n \not\equiv 3 \pmod{4}$ . If  $m \equiv 0 \pmod{2}$ ,  $W_{m,n}$  is cordial if and only if  $n \not\equiv 2 \pmod{4}$ .

Proof: Suppose  $m = 2k + 1$  and  $n \not\equiv 3 \pmod{4}$ . Take  $H = T_{2k} = H_1$ ,  $G = W_n$  and  $G_1 = C_n$ . Then  $(G, H) = W_{m,n}$ . Since  $W_n$  is cordial by (9),  $W_{m,n}$  is cordial by Theorem 1.

Suppose  $m = 2k$  and  $n \not\equiv 2 \pmod{4}$ . Take  $H = T_{2k} = H_1$  and  $G = C_n = G_1$ . Then  $(G, H) = W_{m,n}$ . Since  $C_n$  is cordial by (6),  $W_{m,n}$  is cordial by Theorem 1.

Suppose  $W_{m,n}$  is cordial for some  $m \equiv 1 \pmod{2}$  and  $n \equiv 3 \pmod{4}$ . Consider the one for which  $m$  is minimum, and any cordial labelling of it. Note that  $m \geq 3$  since  $W_n$  is not cordial according to (9). Suppose two vertices not on  $C_n$  have opposite labels. Then  $W_{m,n} = (G, H)$  where  $H = H_1$  consists of these two vertices,  $G = W_{m-2,n}$  and  $G_1 = C_n$ . Since  $W_{m,n}$  is assumed to be cordial, so is  $W_{m-2,n}$  by Theorem 1. This contradicts the minimality assumption.

It follows that all vertices of  $W_{m,n}$  not on  $C_n$  have the same label. Note that  $m + n$  is even. Hence, there are  $m$  more vertices on  $C_n$  with label 1 than those with label 0, so that among the edges not on  $C_n$ , there are  $m^2$  more with label 1 than those with label 0. It follows that there are exactly  $\frac{1}{2}(n - m^2)$  edges with label 1 on  $C_n$ . However,  $\frac{1}{2}(n - m^2) \equiv 1 \pmod{2}$  and  $C_n$  is Eulerian. This contradicts (1). The case  $m \equiv 0 \pmod{2}$  and  $n \equiv 2 \pmod{4}$  can be dealt with in the same way. ■

The following result includes both (3) and (4) as special cases.

**Theorem 6.** A complete  $k$ -partite graph is cordial if and only if the number of parts with an odd number of vertices is at most 3.

Proof: A complete  $k$ -partite graph with no odd parts is clearly cordial, as we can assign the label 0 to exactly half of the vertices in each part. If it has 1, 2 or 3 odd parts, let  $G$  be a complete graph consisting of a single vertex in each odd part and  $H$  be the graph induced by the remaining vertices. Each  $G_i$  consists of one vertex in  $G$ , and the corresponding  $H_i$  consists of all vertices of  $H$  not in the same part of the vertex in  $G_i$ . Then the complete  $k$ -partite graph is equal to  $(G, H)$ . By (3),  $K_n$  is cordial if  $n \leq 3$ . Hence, our graph is cordial by Theorem 1.

Suppose there is a cordial complete  $k$ -partite graph with at least 4 odd parts. Consider one with the smallest number of vertices, and any cordial labelling of it. According to (3),  $K_n$  is not cordial if  $n \leq 4$ . Hence, the graph has at least