2. Edge-magicness of \((p, p)\)-graphs

In this section, we consider the edge-magicness of a multigraph whose order and size are equal.

**Theorem 2.1:** Let \( G \) be a \((p, p)\)-graph. \( G \) is edge-magic if and only if \( G \) is isomorphic to one of the following graph:

1. \( mK_2[2] \),
2. \( G_1 + mK_2[2] \),
3. \( K_2 + G_2 + mK_2[2] \),

for some \( m \), where \( G_1 \) is isomorphic to \( H_1 \) or \( H_2 \) and \( G_2 \) is isomorphic to \( H_3, H_4, H_5 \) or \( H_6 \) (Figure 2.1).

![Figure 2.1](image)

**Proof:** Suppose \( G = (V, E) \) is edge-magic. Since the order and the size of \( G \) are the same, the labels of edges are distinct. Hence every 1-vertex of \( G \) must be incident to the same edge (a vertex of degree \( d \) is called a \( d \)-vertex). Consequently, \( G \) contains at most two 1-vertices, and if \( G \) contains two 1-vertices then these two vertices induce a component of \( G \) isomorphic to \( K_2 \). Moreover, if \( G' \) is a component of \( G \) with two adjacent 2-vertices, then \( G' \cong K_2[2] \).

**Case 1:** Suppose each component of \( G \) has the same order and size. In this case, \( G \) contains at most one 1-vertex. If \( G \) contains no 1-vertex and \( V^* \) is a component of \( G \), then from

\[
\sum_{v \in V^*} \deg(v) = 2e^*,
\]

where \( e^* \) is the size of \( G[V^*] \), we have \( \deg(v) = 2 \) for all \( v \in V^* \). Because order of \( G[V^*] \) is at least 2, then by last sentence of the first paragraph \( G[V^*] \cong K_2[2] \). Therefore \( G \cong mK_2[2] \), where \( m = \frac{e}{2} \).

Suppose \( G \) contains one 1-vertex. From (2.1), the component, say \( G_1 \), containing the 1-vertex must contain one 3-vertex and a number of 2-vertices. If the 1-vertex is adjacent to the 3-vertex, then \( G_1 \cong H_1 \). If the 1-vertex is adjacent to a 2-vertex, then this 2-vertex must be adjacent to the 3-vertex and \( G_1 \cong H_2 \). Since other components of \( G \) contain no