

On Edge-Magic $(p, 3p - 1)$ -Graphs

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Abstract

A (p, q) -graph G in which the edges are labeled $1, 2, 3, \dots, q$ so that the vertex sums are constant, is called *supermagic*. If the vertex sum modulo p is a constant, then G is called *edge-magic*. Lee, Seah and Tan [12] introduced the concept of *edge-magic graphs*, which is a weaker concept than *supermagic graphs*. In this paper, we investigate *edge-magic $(p, 3p-1)$ -graphs*. Using *edge-splitting extension* on stars, trees, cycles and $(p, p+1)$ -graphs, we find several infinite families of *edge-magic graphs*.

Keywords and phrases: *edge-magic, supermagic, edge-splitting extension, tree, partition, amalgamation.*

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1 Introduction

A (p, q) -graph G in which the edges are labeled $1, 2, 3, \dots, q$, so that the vertex sums are constant, is called *supermagic*. Stewart [22, 23] showed

that the complete graphs K_3 , K_4 and K_5 are not supermagic and for $n > 5$, K_n is supermagic if and only if $n \equiv 0 \pmod{4}$. Hartsfield and Ringel [4] provided some classes of supermagic graphs. Ho and Lee [5] extended the result of Stewart to regular complete k -partite graphs. Recently Shiu, Lam and Cheng [16] considered a class of supermagic graphs which are disjoint union of $K_{3,3}$. Shiu, Lam and Lee [17] proved that the composition of regular supermagic graph with a null graph is always supermagic. In [6] supermagic regular multipartite graphs and supermagic cubes are characterized.

Definition 1. If G is a (p, q) -multigraph in which the edges are labeled $1, 2, 3, \dots, q$ so that the vertex sums are constant, modulo p , then G is called an *edge-magic graph*, or simply an EM graph.

Example 1. Figure 1 shows a non supermagic graph G_1 with 6 vertices and 8 edges that is edge-magic with different constant sums. \square

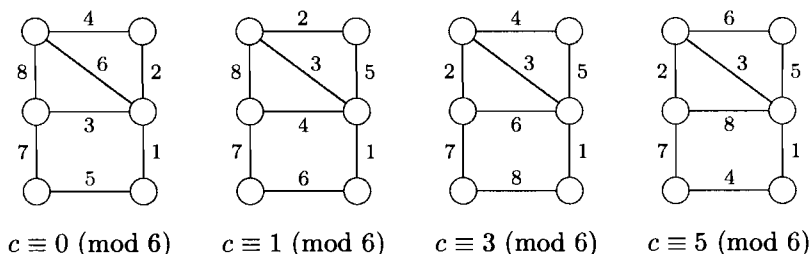


Figure 1: Edge-magic labelings of a graph.

Example 2. Figure 2 shows another graph G_2 with 6 vertices and 8 edges that is edge-magic with different constant sums. \square

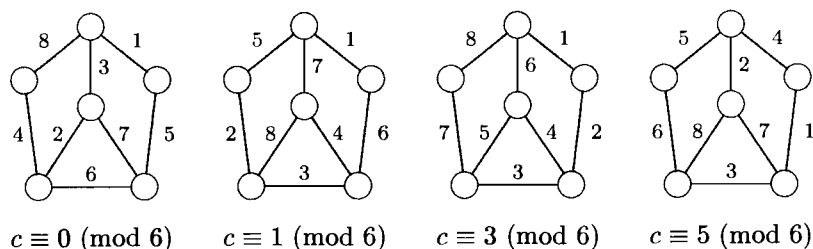


Figure 2: Edge-magic labelings of another graph.

The concept of supermagic graphs was introduced by Stewart [21] in 1966. Edge-magic graphs was introduced by the third author, Seah and Tan in [12]. In [2, 7, 8], other types of magic graph labelings were discussed; but they are not related to this paper.

A necessary condition for a (p, q) -graph to be edge-magic is $q(q+1) \equiv 0 \pmod{p}$. However, there are infinitely many connected graphs such as trees and cycles that satisfy this condition but are not edge-magic.

In [24], we introduced a general construction of multigraphs from a given simple graph. Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of natural numbers. Given a pair (G, f) , where $G = (V, E)$ is a simple graph, and an edge-labeling $f : E(G) \rightarrow \mathbb{N}$, we construct a multigraph $\text{SPE}(G, f)$ by associating to each $e \in E(G)$ a set $P(e)$ of $f(e)$ parallel edges. Notice that $V(\text{SPE}(G, f)) = V(G)$ and $E(\text{SPE}(G, f)) = \bigcup_{e \in E(G)} P(e)$. We call $\text{SPE}(G, f)$ an **edge-splitting extension** of (G, f) , and G the **skeleton** of $\text{SPE}(G, f)$.

Example 3. An edge-labeling f of C_4 and its edge-splitting extension are displayed in Figure 3. □

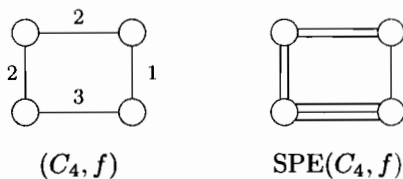


Figure 3: The edge-splitting extension of (C_4, f) .

Finding an EM labeling of a given graph is related to solving a system of linear Diophantine equations. In general, it is difficult to find an EM labeling or a supermagic labeling of a graph. Several classes of graphs had been shown to be EM [10, 11, 12, 13, 14, 15, 17, 19, 20]. We refer the readers to [11, 13] for more conjectures and open problems on EM graphs.

Recently, Wen, Lee and Sun [24] used edge-splitting extension to construct many new supermagic multigraphs. This prompts us to investigate the conditions under which an edge-splitting extension graph is EM. We shall use h to denote the edge labeling of $\text{SPE}(G, f)$. We describe h by naming the set of numbers to be used in $h(P(e))$ for each $e \in E(G)$.

Definition 2. Given a simple (p, q) -graph G , and an integer $q^* > q$, a function $f : E(G) \rightarrow \mathbb{N}$ is said to be q^* -**admissible** if $\text{SPE}(G, f)$ has q^* edges.

We observe that a $(p, kp-1)$ -multigraph satisfies the necessary condition for it to be EM. Consequently, we are interested in finding a graph G

with a $(kp - 1)$ -admissible f such that $\text{SPE}(G, f)$ is EM. If S is a set, we use $k \times S$ to denote the multiset composed of k copies of S . Since $\{1, 2, \dots, kp - 1\} \equiv (k - 1) \times \{0\} \cup k \times \{1, 2, \dots, p - 1\} \pmod{p}$, we will label the edges of $\text{SPE}(G, f)$ with the multiset $(k - 1) \times \{0\} \cup k \times \{1, 2, \dots, p - 1\}$. We observe that even if p divides $q(q + 1)$, there is no guarantee that a (p, q) -graph is EM.

Theorem 1.1 *Let G be a graph with $k + 1$ pendant edges e_i , and let f be a $(kp - 1)$ -admissible function such that $f(e_i) = 1$ for each i . Then $\text{SPE}(G, f)$ cannot be EM.*

Proof. To be EM, the vertex sums must be constant. Thus each of the $k + 1$ pendant edge must be assign the same edge-label, which is impossible with the multiset $(k - 1) \times \{0\} \cup k \times \{1, 2, \dots, p - 1\}$. Hence $\text{SPE}(G, f)$ cannot be EM. \square

Example 4. Consider $k = 2$. The two edge-labelings f_1 and f_2 of the $(8, 9)$ -graph G depicted in Figure 4 are both 15-admissible. However, $\text{SPE}(G, f_1)$ is EM, but $\text{SPE}(G, f_2)$ is not. \square

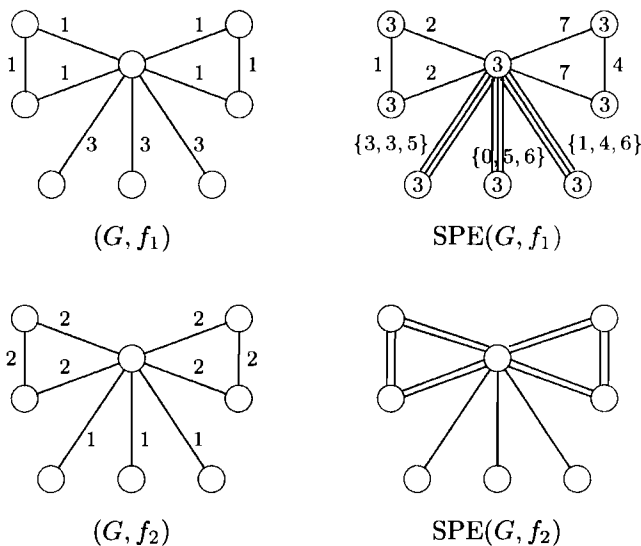


Figure 4: Two different edge-splitting extensions of the same graph.

Lee, Chen and Tong [9] investigated $(p, 2p - 1)$ -graphs that are EM. In this paper, we continue the study with $(p, 3p - 1)$ -graphs. More specifically, we study the EM labeling of $\text{SPE}(G, f)$ with $2 \times \{0\} \cup 3 \times \{1, 2, \dots, p - 1\}$, where G is a graph of order p , and f is a $(3p - 1)$ -admissible function.

2 Preliminaries

Let the edges of G be e_1, e_2, \dots, e_q . To describe an EM labeling h , it suffices to name the edge labels for each $P(e_i)$. Define $\sigma_i = \sum_{e \in P(e_i)} h(e)$. Then

$$\sum_{i=1}^q \sigma_i = 3(1 + 2 + \dots + (p-1)) = \frac{3p(p-1)}{2}.$$

It follows that

$$\sum_{i=1}^q \sigma_i \equiv \begin{cases} 0 \pmod{p} & \text{if } p \text{ is odd,} \\ \frac{p}{2} \pmod{p} & \text{if } p \text{ is even.} \end{cases}$$

In particular, when p is odd, we would attempt to find an EM labeling with $\sigma_i \equiv 0 \pmod{p}$ for all i .

Theorem 2.1 *Let G be a graph of odd order p with a $(3p-1)$ -admissible f such that $f(e)$ is even for every $e \in E(G)$, with the exception of either none or exactly two edges. Then $\text{SPE}(G, f)$ is EM.*

Proof. Arrange the edge labels as follows:

$$1, p-1, 1, p-1, 1, p-1, 2, p-2, \dots, (p-1)/2, (p+1)/2.$$

For $i = 1, 2, \dots, q$, assign the next unused $f(e_i)$ or $f(e_i) - 1$ labels to $P(e_i)$, depending on whether $f(e_i)$ is even or odd, respectively. The two zeros will be assigned to the $P(e_i)$ s that are unfilled. The result is an EM labeling with vertex sum 0. \square

Example 5. Figures 5 and 6 display, respectively, the EM labeling of the edge-splitting extension of a tree and a unicyclic graph as they are described in Theorem 2.1. \square

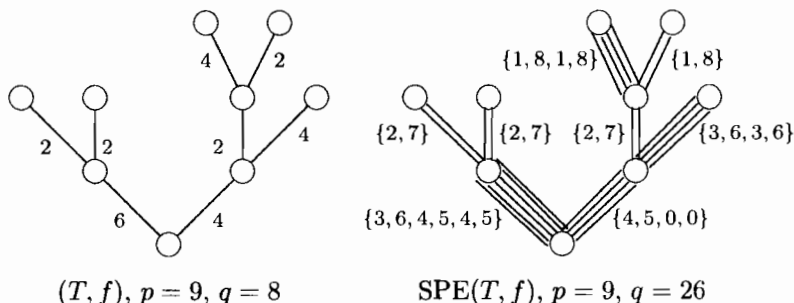
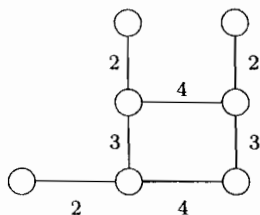
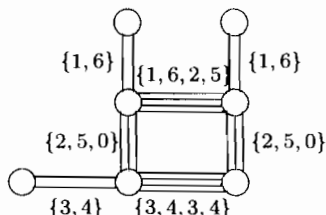


Figure 5: An EM labeling of $\text{SPE}(T, f)$, where $f(e)$ is even.



$(G, f), p = 7, q = 8$



$\text{SPE}(G, f), p = 7, q = 8$

Figure 6: Labeling of $\text{SPE}(G, f)$, where $f(e)$ is odd for exactly two edges.

3 The Edge-Splitting Extension of Trees

Recall that the star $\text{St}(n)$ is a tree of diameter 2 with $n + 1$ vertices.

Theorem 3.1 *The edge-splitting extension $\text{SPE}(\text{St}(n), f)$ is EM for any $(3n + 2)$ -admissible f if and only if $n = 1, 2, 3$.*

Proof. The case of $n = 1$ is obvious. There are four non-isomorphic 8-admissible functions f when $n = 2$. They are listed in the following table along with the EM labelings of their respective edge-splitting extensions.

$E(\text{St}(2))$		$E(\text{SPE}(\text{St}(2), f))$	
$f(e_1)$	$f(e_2)$	$h(P(e_1))$	$h(P(e_2))$
1	7	{0}	{0, 1, 1, 1, 2, 2, 2}
2	6	{1, 2}	{0, 0, 1, 1, 2, 2}
3	5	{0, 1, 2}	{0, 1, 1, 2, 2}
4	4	{0, 0, 1, 2}	{1, 1, 2, 2}

The EM labelings of $\text{SPE}(\text{St}(3))$ with constant vertex sum 2 (mod 4) for all 11-admissible functions f are listed below.

$E(\text{St}(3))$			$E(\text{SPE}(\text{St}(3), f))$		
$f(e_1)$	$f(e_2)$	$f(e_3)$	$h(P(e_1))$	$h(P(e_2))$	$h(P(e_3))$
1	1	9	{2}	{2}	{0, 0, 1, 1, 1, 2, 3, 3, 3}
1	2	8	{2}	{0, 2}	{0, 1, 1, 1, 2, 3, 3, 3}
1	3	7	{2}	{0, 0, 2}	{1, 1, 1, 2, 3, 3, 3}
1	4	6	{2}	{0, 0, 1, 1}	{1, 2, 2, 3, 3, 3}
1	5	5	{2}	{0, 0, 1, 2, 3}	{1, 1, 2, 3, 3}
2	2	7	{0, 2}	{0, 2}	{1, 1, 1, 2, 3, 3, 3}
2	3	6	{0, 2}	{0, 1, 1}	{1, 2, 2, 3, 3, 3}
2	4	5	{0, 2}	{0, 1, 2, 3}	{1, 1, 2, 3, 3}
3	3	5	{0, 0, 2}	{1, 2, 3}	{1, 1, 2, 3, 3}
3	4	4	{0, 0, 2}	{1, 1, 1, 3}	{2, 2, 3, 3}

For $n \geq 4$, consider f with $f(e_i) = 1$ if $i < n$, and $f(e_n) = 2n + 3$. Because of Theorem 1.1, we only need to look at $n = 4$. The vertex sums of the four pendant vertices must be the same:

$$c = h(e_1) = h(e_2) = h(e_3) \equiv \sigma_4 \pmod{5}.$$

The vertex sum at the center is $4c \equiv c \pmod{5}$; hence $c \equiv 0 \pmod{5}$. This means $h(e_1) = h(e_2) = h(e_3) = 0$, which is impossible because only two zeros are available. \square

Theorem 3.2 *Let T be a tree of odd order p with a $(3p - 1)$ -admissible f such that $|f^{-1}(1) \cup f^{-1}(2)| \leq 1$. Then $\text{SPE}(G, f)$ is EM.*

Proof. Assume $p = 2k + 1$. For $1 \leq i \leq k$, define $S_{2i-1} = \{i, i, p - 2i\}$ and $S_{2i} = \{p - i, p - i, 2i\}$. Most of the edges in G will split into three edges, so we would try to assign as many S_j 's as possible to these edges, and then regroup the remaining labels so that the edge sum in each modified subset is still 0. For example, we can regroup $S_{2k-3} = \{k - 1, k - 1, 3\}$, $S_{2k-2} = \{k + 2, k + 2, 2k - 2\}$, $S_{2k-1} = \{k, k, 1\}$ and $S_{2k} = \{k + 1, k + 1, 2k\}$ into $X = \{k - 1, k - 1, k + 2, k + 2\}$, $Y = \{k, k, k + 1, k + 1\}$ and $Z = \{3, 2k - 2, 1, 2k\}$. In some occasions, we use $S_{2k-1} \cup S_{2k} = Y \cup \{1, 2k\}$. There are ten cases, we identify them by the notation $1^{n_1}2^{n_2} \dots$, where n_j is the number of edges e with $f(e) = j$. In each case, we list below the subsets to be used as $h(P(e_i))$ that yield $\sigma_i \equiv 0 \pmod{p}$. \square

$3^{2k-2}4^2$	$S_1, S_2, \dots, S_{2k-2}, S_{2k-1} \cup \{0\}, S_{2k} \cup \{0\}$
$3^{2k-1}5^1$	$S_1, S_2, \dots, S_{2k-1}, S_{2k} \cup \{0, 0\}$
$2^13^{2k-4}4^3$	$S_1, S_2, \dots, S_{2k-4}, X, Y, Z, \{0, 0\}$
$2^13^{2k-3}4^15^1$	$S_1, S_2, \dots, S_{2k-3}, S_{2k-2} \cup \{0, 0\}, Y, \{1, 2k\}$
$2^13^{2k-2}6^1$	$S_1, S_2, \dots, S_{2k-2}, S_{2k-1} \cup S_{2k}, \{0, 0\}$
$1^13^{2k-5}4^4$	$S_1, S_2, \dots, S_{2k-5}, S_{2k-4} \cup \{0\}, X, Y, Z, \{0\}$
$1^13^{2k-4}4^25^1$	$S_1, S_2, \dots, S_{2k-5}, S_{2k-4}, X, Y, Z \cup \{0\}, \{0\}$
$1^13^{2k-3}4^16^1$	$S_1, S_2, \dots, S_{2k-3}, S_{2k-2} \cup S_{2k-1}, S_{2k} \cup \{0\}, \{0\}$
$1^13^{2k-3}5^2$	$S_1, S_2, \dots, S_{2k-5}, S_{2k-4}, S_{2k-3} \cup \{k, k + 1\}, S_{2k-2} \cup \{k, k + 1\}, \{1, 2k, 0\}, \{0\}$
$1^13^{2k-2}7^1$	$S_1, S_2, \dots, S_{2k-3}, S_{2k-2}, S_{2k-1} \cup S_{2k} \cup \{0\}, \{0\}$

Example 6. Some cases are vacuous when $p = 3, 5$; we have all ten cases starting with $p = 7$. The non-vacuous cases are listed below. \square

$p = 3, q = 8$	$h(P(e_i))$
4^2	$\{1, 1, 1, 0\}, \{2, 2, 2, 0\}$
3^15^1	$\{1, 1, 1\}, \{2, 2, 2, 0, 0\}$
2^16^1	$\{1, 1, 1, 2, 2, 2\}, \{0, 0\}$
1^17^1	$\{1, 1, 1, 2, 2, 2, 0\}, \{0\}$

$p = 5, q = 14$	$h(P(e_i))$
$3^2 4^2$	$\{1, 1, 3\}, \{4, 4, 2\}, \{2, 2, 1, 0\}, \{3, 3, 4, 0\}$
$3^3 5^1$	$\{1, 1, 3\}, \{4, 4, 2\}, \{2, 2, 1\}, \{3, 3, 4, 0, 0\}$
$2^1 4^3$	$\{1, 4, 1, 4\}, \{2, 3, 2, 3\}, \{3, 2, 1, 4\}, \{0, 0\}$
$2^1 3^1 4^1 5^1$	$\{1, 1, 3\}, \{4, 4, 2, 0, 0\}, \{2, 3, 2, 3\}, \{1, 4\}$
$2^1 3^2 6^1$	$\{1, 1, 3\}, \{4, 4, 2\}, \{2, 2, 1, 3, 3, 4\}, \{0, 0\}$
$1^1 4^2 5^1$	$\{1, 4, 1, 4\}, \{2, 3, 2, 3\}, \{3, 2, 1, 4, 0\}, \{0\}$
$1^1 3^1 4^1 6^1$	$\{1, 1, 3\}, \{4, 4, 2, 2, 2, 1\}, \{3, 3, 4, 0\}, \{0\}$
$1^1 3^1 5^2$	$\{1, 1, 3, 2, 3\}, \{4, 4, 2, 2, 3\}, \{1, 4, 0\}, \{0\}$
$1^1 3^2 7^1$	$\{1, 1, 3\}, \{4, 4, 2\}, \{2, 2, 1, 3, 3, 4, 0\}, \{0\}$

$p = 7, q = 20$	$h(P(e_i))$
$3^4 4^2$	$\{1, 1, 5\}, \{6, 6, 2\}, \{2, 2, 3\}, \{5, 5, 4\}, \{3, 3, 1, 0\}, \{4, 4, 6, 0\}$
$3^5 5^1$	$\{1, 1, 5\}, \{6, 6, 2\}, \{2, 2, 3\}, \{5, 5, 4\}, \{3, 3, 1\}, \{4, 4, 6, 0, 0\}$
$2^1 3^2 4^3$	$\{1, 1, 5\}, \{6, 6, 2\}, \{2, 5, 2, 5\}, \{3, 4, 3, 4\}, \{3, 4, 1, 6\}, \{0, 0\}$
$2^1 3^3 4^1 5^1$	$\{1, 1, 5\}, \{6, 6, 2\}, \{2, 2, 3\}, \{5, 5, 4, 0, 0\}, \{3, 4, 3, 4\}, \{1, 6\}$
$2^1 3^4 6^1$	$\{1, 1, 5\}, \{6, 6, 2\}, \{2, 2, 3\}, \{5, 5, 4\}, \{3, 3, 1, 4, 4, 6\}, \{0, 0\}$
$1^1 3^1 4^4$	$\{1, 1, 5\}, \{6, 6, 2, 0\}, \{2, 5, 2, 5\}, \{3, 4, 3, 4\}, \{3, 4, 1, 6\}, \{0\}$
$1^1 3^2 4^2 5^1$	$\{1, 1, 5\}, \{6, 6, 2\}, \{2, 5, 2, 5\}, \{3, 4, 3, 4\}, \{3, 4, 1, 6, 0\}, \{0\}$
$1^1 3^3 4^1 6^1$	$\{1, 1, 5\}, \{6, 6, 2\}, \{2, 2, 3\}, \{5, 5, 4, 3, 3, 1\}, \{4, 4, 6, 0\}, \{0\}$
$1^1 3^3 5^2$	$\{1, 1, 5\}, \{6, 6, 2\}, \{2, 2, 3, 3, 4\}, \{5, 5, 4, 3, 4\}, \{1, 6, 0\}, \{0\}$
$1^1 3^4 7^1$	$\{1, 1, 5\}, \{6, 6, 2\}, \{2, 2, 3\}, \{5, 5, 4\}, \{3, 3, 1, 4, 4, 6, 0\}, \{0\}$

When $p = 2k$, preliminary data suggest that it is possible to partition the edge labels so that $\sigma_i = k$ for each i ; consequently, $\text{SPE}(T, f)$ would be EM if the degrees of T are all odd. The partitions appear to be rather complicated, and seem to depend on $k \pmod{3}$. We leave the following open problem for further investigation.

Open Problem 1 *Let T be a tree of even order p such that all its degrees are odd. Then $\text{SPE}(G, f)$ is EM for any $(3p - 1)$ -admissible function $f : E(T) \rightarrow \mathbb{N}$ with $|f^{-1}(1) \cup f^{-1}(2)| \leq 1$.*

4 The Edge-Splitting Extension of Cycles

Let the vertices of C_n be v_1, v_2, \dots, v_n so that $e_i = v_i v_{i+1}$. Since e_i is incident to e_{i-1} and e_{i+1} , if $\text{SPE}(C_n, f)$ is EM, we must have

$$\sigma_1 \equiv \sigma_3 \equiv \dots \pmod{n} \quad \text{and} \quad \sigma_2 \equiv \sigma_4 \equiv \dots \pmod{n}.$$

If n is odd, we also have $\sigma_n \equiv \sigma_2 \pmod{n}$, hence σ_i is a constant \pmod{n} .

Theorem 4.1 *The graph $\text{SPE}(C_n, f)$ is EM for any $f : E(C_n) \rightarrow \mathbb{N}$ with the property that $f(e^*) = 2$ for a specific edge e^* and $f(e) = 3$ for all $e \neq e^*$,*

Proof. When $n = 2k + 1$, assume $e^* = e_n$, and partition the edge labels into $(\bigcup_{i=1}^{2k} S_i) \cup \{0, 0\}$, where S_i is defined in Theorem 3.2. Then $\sigma_i \equiv 0 \pmod{n}$ for all i , thereby producing an EM labeling of $\text{SPE}(C_n, f)$ with constant vertex sum 0.

For $n = 2k$, our goal is to find σ_i such that

$$\sigma_i \equiv \begin{cases} a \pmod{2k} & \text{if } i \text{ is odd,} \\ b \pmod{2k} & \text{if } i \text{ is even,} \end{cases}$$

so that the vertex sum is always $a + b$. This requires

$$k(a + b) = \sum_{i=1}^{2k} \sigma_i = 3 \sum_{j=1}^{2k-1} j = 3k(2k - 1),$$

which in turn implies that

$$a + b = 3(2k - 1) \equiv -3 \pmod{2k}.$$

We now describe a labeling with $a = k - 1$ and $b = k - 2$.

When k is odd, define for each i ,

$$x_i \equiv \begin{cases} k - 1 - 2i \pmod{2k} & \text{if } i \text{ is even,} \\ k - 2 - 2i \pmod{2k} & \text{if } i \text{ is odd.} \end{cases}$$

When k is even, define x_i the same way if $0 \leq i \leq k - 1$; but define

$$x_i \equiv \begin{cases} k - 2 - 2i \pmod{2k} & \text{if } i \text{ is even,} \\ k - 1 - 2i \pmod{2k} & \text{if } i \text{ is odd,} \end{cases}$$

if $k \leq i \leq 2k - 1$. Let $T_i = \{i, i, x_i\}$ if $x_i \neq 0$, and $T_i = \{i, i\}$ if $x_i = 0$. It is not difficult to verify that the nonzero values among x_i 's are distinct, hence $\bigcup_{i=0}^{2k-1} T_i$ forms a partition of the edge labels that meets the requirement. \square

Example 7. Let f be a function described in Theorem 4.1. The EM labeling of $\text{SPE}(C_n, f)$ for $n = 4, 6, 8, 10$ are described below. \square

$n = 4$	σ
0 0 1	1
1 1 2	0
2 2	0
3 3 3	1

$n = 6$	σ
0 0 2	2
1 1 5	1
2 2 4	2
3 3 1	1
4 4	2
5 5 3	1

$n = 8$	σ
0 0 3	3
1 1	2
2 2 7	3
3 3 4	2
4 4 2	2
5 5 1	3
6 6 6	2
7 7 5	3

$n = 10$	σ
0 0 4	4
1 1 1	3
2 2	4
3 3 7	3
4 4 6	4
5 5 3	3
6 6 2	4
7 7 9	3
8 8 8	4
9 9 5	3

Example 8. Let f be a function described in Theorem 4.1. An EM labeling of $\text{SPE}(C_3, f)$ is obtained when the edges are labeled $\{1, 1, 1\}$, $\{2, 2, 2\}$, $\{0, 0\}$. Likewise, labeling the edges of $\text{SPE}(C_5, f)$ with $\{1, 1, 3\}$, $\{4, 4, 2\}$, $\{2, 2, 1\}$, $\{3, 3, 4\}$, $\{0, 0\}$ yields an EM labeling. All the EM labeling has 0 vertex sum. \square

Theorem 4.2 *Assume $f : E(C_{2k+1}) \rightarrow \mathbb{N}$ has the property that $f(e_{2k+1}) = 2$, and for $1 \leq j \leq k$, $f(e_{2j-1}) = x$ and $f(e_{2j}) = y$. Then $\text{SPE}(C_{2k+1}, f)$ is always EM when $\{x, y\} = \{3, 3\}, \{2, 4\}$; but when $\{x, y\} = \{1, 5\}$, $\text{SPE}(C_{2k+1}, f)$ is EM if and only if $k \leq 3$.*

Proof. When $x = y = 3$, label $P(e_{2k+1})$ with $\{0, 0\}$, and for $i < 2k + 1$, label $P(e_i)$ with S_i that was described in the proof of Theorem 3.2. When $\{x, y\} = \{2, 4\}$, the result follows immediately from Theorem 2.1. Since all the edges e with $f(e) = 1$ must be assigned the same edge label, it is clear that we need $k \leq 3$ when $\{x, y\} = \{1, 5\}$. We now describe the labels for $P(e)$ when $k = 1, 2, 3$:

$$k = 1 : \quad \{1\}, \{0, 1, 2, 2, 2\}, \{0, 1\}.$$

$$k = 2 : \quad \{1\}, \{0, 2, 3, 3, 3\}, \{1\}, \{2, 2, 4, 4, 4\}, \{0, 1\}.$$

$$k = 3 : \quad \{1\}, \{0, 2, 3, 5, 5\}, \{1\}, \{0, 3, 4, 4, 4\}, \{1\}, \{2, 3, 5, 6, 6\}, \{2, 6\}.$$

In each case, we always have $\sigma_i = 1$, which yields an EM labeling with vertex sum 2. \square

What happens when $n = 2k$? The cases are similar to that of Theorem 4.2. However σ_{2j-1} and σ_{2j} need not be the same. In fact, it can be shown that $\sigma_1 + \sigma_2 \equiv 1 \pmod{2}$. The cases of $k = 2, 3$ are easy, but larger k requires more analysis. We once again invite the readers to study the possibilities.

5 The Edge-Splitting Extension of $(p, p + 1)$ -Graphs

In this section, we study the edge-splitting extension of $(p, p + 1)$ -graphs where $f(e^*) = p - 1$ for a specific edge e^* and $f(e) = 2$ whenever $e \neq e^*$.

Theorem 5.1 *Let G be a $(p, p + 1)$ -graph of odd order. Then $\text{SPE}(G, f)$ is EM for any $f : E(C_n) \rightarrow \mathbb{N}$ with the property that $f(e^*) = p - 1$ for a specific edge e^* and $f(e) = 2$ for all $e \neq e^*$.*

Proof. This result follows from Theorem 2.1 because $p - 1$ is even. \square

Example 9. Figure 7 illustrates the EM labelings on a $(p, p + 1)$ -graph of odd order as they are described in Theorems 5.1 and 2.1. \square

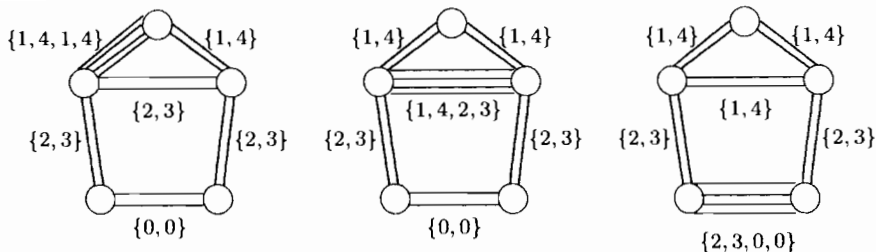


Figure 7: Three EM labelings of a $(p, p + 1)$ -graph of odd order.

The solution is more complicated when p is even. We study three families of $(p, p + 1)$ -graphs:

- Assume the vertices of C_n are v_1, v_2, \dots, v_n . The **theta graph** $C_n(r)$ is obtained by adding the chord v_1v_r . The skeleton of the edge-splitting extension graphs in Figure 7 is $C_5(3)$. The graphs displayed in Figure 8 are $C_6(3)$ and $C_6(4)$.

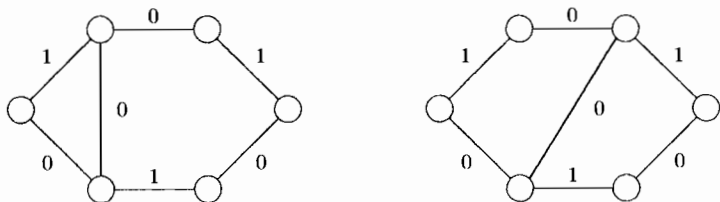


Figure 8: The theta graphs $C_6(3)$ and $C_6(4)$.

- The **one-point union** of C_m and C_n , denoted $C(m, n)$, is the graph obtained by identifying one vertex of C_m to another vertex in C_n ; that is, $C(m, n) = \text{Amal}(C_m, C_n, (u, v))$. Note that for $p = m + n - 1$ to be even, m and n must have different parity. See Figure 9.

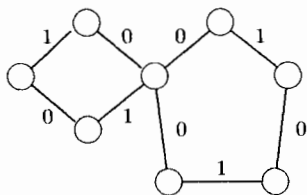


Figure 9: The one-point union $C(4, 5)$.

- The **dumbbell graph** $D(m, n)$ is obtained from C_m and C_n by connecting a vertex from each cycle with an edge. Note that for $p = m + n$ to be even, m and n must have the same parity. See Figure 10.

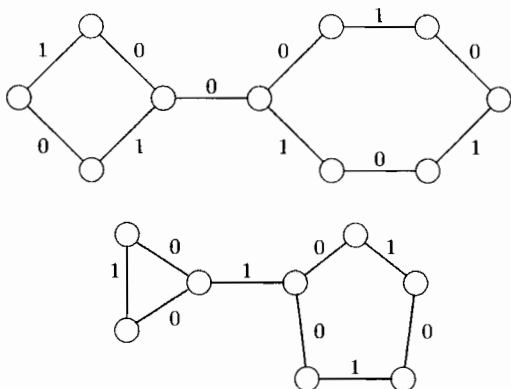


Figure 10: The dumbbell graphs $D(4, 6)$ and $D(3, 5)$.

All three families share a common property. We can label the edges of all these graphs with 0 and 1 such that the vertex sum is 1. See Figures 7–9 for some illustrations. In addition, we claim that the edges can be labeled in such a way that the numbers of 0- and 1-edges are $k + 1$ and k respectively:

- For $C_n(r)$, label the edges on the cycle alternately with 0 and 1, and the chord with 0. See Figure 8.
- For $C(m, n)$, one cycle is odd, the other is even. Label their edges alternately with 0 and 1 such that the two adjacent 0-edges on the odd cycles are incident to the even cycle. See Figure 9.
- For $D(m, n)$, the two cycles must be both even or both odd. Label the edges of two cycles alternately with 0 and 1. If m and n are even, the bridge connecting the two cycles must be an 0-edge. If both m and n are odd, use the bridge to connect the two pairs of adjacent 0-edges from the two cycles; and label the bridge with 1. See Figure 10.

More interestingly, the fact that their edge-splitting extensions are EM can be established with the same proof!

Theorem 5.2 *Let G be $C_p(r)$, $C(m, p + 1 - m)$ or $D(m, p - m)$, where p is even. Then $\text{SPE}(G, f)$ is EM for any $f : E(C_n) \rightarrow \mathbb{N}$ with the property that $f(e^*) = p - 1$ for a specific edge e^* and $f(e) = 2$ for all $e \neq e^*$.*

Proof. Let $p = 2k$. Define $S_i = \{i, p - i\}$ and $T_j = \{j, p + 1 - j\}$, reduced modulo p whenever necessary, so that their edge sum is 0 and 1 respectively. Let $R = \{1, 2, \dots, k - 1, k + 1, \dots, p - 1\}$; note that the edge sum of R is 0. Find an edge-labeling of G with 0 and 1 with constant vertex sum 1, and the property that the numbers of 0- and 1-edges are $k + 1$ and k respectively. We now describe the labeling h on $E(\text{SPE}(G, f))$.

- If e^* is an 0-edge, label $P(e^*)$ with $\{0\} \cup R$, and for $e \neq e^*$, label $P(e)$ with S_i and T_j , where $1 \leq i, j \leq k$, depending on whether e is an 0-edge or an 1-edge, respectively.
- If e^* is an 1-edge, label $P(e^*)$ with $\{1\} \cup R$, and for $e \neq e^*$, label $P(e)$ with S_i and T_j , where $0 \leq i \leq k$ and $2 \leq j \leq k$, depending on whether e is an 0-edge or an 1-edge, respectively.

It is straightforward to verify that the resulting labeling of $\text{SPE}(G, f)$ is EM with vertex sum 1. \square

Example 10. We depict in Figure 11 the two EM labelings described in Theorem 5.2 on $\text{SPE}(C_6(3), f)$ for two different f . These labelings are based on the edge-labeling of $C_6(3)$ given in Figure 8. \square

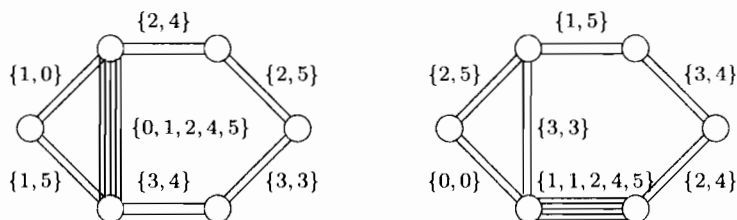


Figure 11: Two EM labelings of $C_6(3)$.

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