THE SUPER EDGE-GRACEFULNESS OF TWO INFINITE FAMILIES OF TREES

SIN-MIN LEE, HUGO SUN, WANDI WEI*, YIHUI WEN, AND PAUL YIU

ABSTRACT. For a positive integer q, let $L(q)$ be the set of k integers, smallest in absolute value, and symmetric about 0. A connected, simple (p,q)-graph $G = (V, E)$ is said to be super edge-graceful if there is a bijection $f : E \longrightarrow L(q)$ inducing a bijection $f^* : V \longrightarrow L(p)$ via $f^*(u) = \sum_{\{u,v\}\in E} f(u,v)$. Let $T(n;(a_1, a_2, \ldots, a_n))$ be the tree obtained by amalgamating the path P_n at each vertex u_t, $t = 1, 2, \ldots, n$, with a path of length a_t. We validate a conjecture of Lee and Wei that the trees $T(2m+3;(02^{2m+1}0))$ and $T(2m+2;(02^{2m}1))$ are super edge-graceful by giving in each case a large lower bound, exponential in m, of the number of super edge-graceful labelings of the tree.

Key words and phrases: edge-graceful, super edge-graceful, trees, graph labelings, amalgamation, S-triplet partition.

AMS 2000 MSC : 05C78, 05C25

1. INTRODUCTION

In this article we study the super edge-gracefulness of some families of graphs and give a large lower bound of the number of super edge-graceful labelings for each of them. The notion of a super edge-graceful graph was introduced by J. Mitchem and A. Simoson [11]. Let $\mathbb{Z}$ be the set of integers. For $a,\ b \in \mathbb{Z}$, let

$$\mathbb{Z}[a,b] := \{z \in \mathbb{Z} :\ a \le z \le b\}.$$

In an obvious way we define its counterparts when $\mathbb{Z}$ is replaced by $\mathbb{N}$ (the set of natural numbers), $\mathbb{N}_0$ (nonnegative integers), $\mathbb{O}$ (odd integers), and $\mathbb{E}$ (even integers). Note that some values of a, b may make some of these sets the empty set.

For any $k \in \mathbb{N}$, let

$$L(k) = \begin{cases} \mathbb{Z}\left[-\frac{k-1}{2}, \frac{k-1}{2}\right], & \text{when } k \text{ is odd,} \\ \mathbb{Z}\left[-\frac{k}{2}, \frac{k}{2}\right] \setminus \{0\}, & \text{when } k \text{ is even.} \end{cases}$$

Thus, $L(k)$ consists of the k integers, smallest in absolute value, *symmetric* with respect to 0.

By a (p,q)-graph we mean a connected, simple graph with p vertices and q edges. Such a graph $G=(V,E)$ is said to be **super edge-graceful, SEG** for short, if there exists a bijection $f : E \to L(q)$ such that the induced mapping $f^* : V \to \mathbb{Z}$ defined by

$$f^*(u) = \sum_{\{u,v\}\in E} f(u,v) \tag{1}$$

is a bijection onto $L(p)$. Such a bijection f is called an **SEG labeling** of G, $f(e)$ $(e \in E)$ the **label of** e under f, and $f^*(u)$ $(u \in V)$ the **label of** u under f.

Example 1. If $n = 2k$, the path P_{n+1} (with n edges and $n+1$ vertices) is super edge-graceful. Here is an SEG labeling. We label the first half the vertices beginning with k, and decreasing steadily by 1 to the "central" vertex 0. The edge labels, beginning with k and -1, decrease steadily by 1 for every two consecutive edges, to immediately preceding the vertex 0. The labels in the remaining half is the reflection of this in 0, with reversal of sign in every vertex and edge label. Figures 1 and 2 show this labeling respectively for even and odd values of k. Note that by joining the two end vertices (with labels k and $-k$) with an edge labeled 0, we obtain a super edge-graceful labeling of the cycle C_{2k+1}.

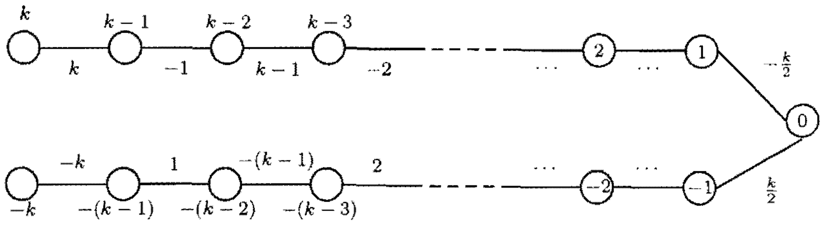

FIGURE 1. SEG labeling of P_{2k+1} for even k

On the other hand, it is easy to check that P_4 and P_6 are not super edge-graceful. More generally, trees of order 4 and 6 are not super edge-graceful; see [3]. Figure 3, however, shows that the P_8 and P_{10} are.

Example 2. It is shown in [6] that the five trees of order 8 in Figure 4 are not super edge-graceful.

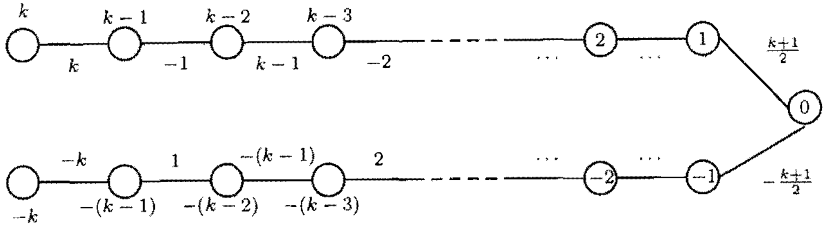

FIGURE 2. SEG labeling of P_{2k+1} for odd k

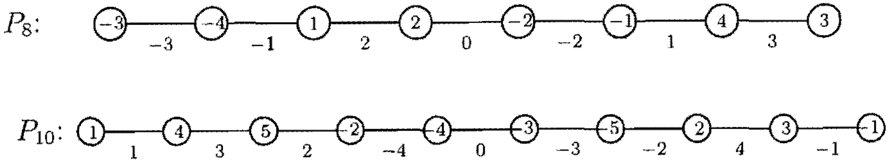

FIGURE 3. SEG labelings of P_8 and P_{10}

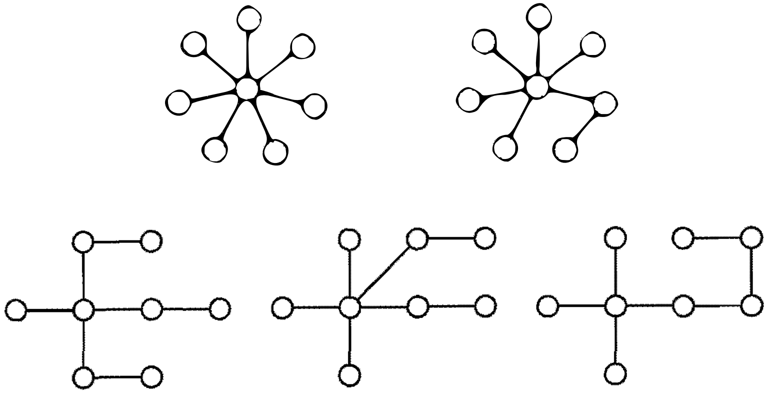

FIGURE 4. Five trees which are not super edge-graceful

The concept of an SEG graph is closely related to the one of an edge-graceful graph introduced by S. P. Lo [10]. A (p,q)-graph $G=(V,E)$ is said to be **edge-graceful** if there exists a bijection $f' : E \to \mathbb{N}[1,q]$ such that its induced mapping $(f')^* : V \to \mathbb{N}$ defined by

$$(f')^*(u) = \sum_{\{u,v\}\in E} f'(u,v) \mod p$$

is a bijection onto $\mathbb{N}_0[0, p-1]$, where $a \mod p$ stands for the least nonnegative remainder of $a \in \mathbb{Z}$ divided by p. Mitchem and Simoson [11] have shown that a super edge-graceful (p,q)-graph is edge-graceful if

$$q \equiv \begin{cases} -1 \pmod{p} & \text{when } q \text{ is even,} \\ 0 \pmod{p} & \text{when } q \text{ is odd.} \end{cases}$$

Therefore, any super edge-graceful tree of odd order is edge-graceful. It is also known that any tree of odd order with diameter at most five is edge-graceful [8, 9]. It is not true, however, that every SEG graph is edge-graceful; see [1, p.124].

2. THE TREE $T(n; (a_1, a_2, \ldots, a_n))$

Given $n \geq 3$ and a list of nonnegative integers $(a_1, a_2, \ldots, a_n)$, we consider, following [4], the tree $T(n; (a_1, a_2, \ldots, a_n))$ obtained by amalgamating P_n (with n vertices) at each vertex u_t, $t = 1, \ldots, n$, with a path of length a_t.

If $a_2 = \cdots = a_{n-1} = a$, we simply denote the tree by $T(n; (a_1 a^{n-2} a_n))$. For example, Figure 6 shows the tree $T(n+2; 02^n0)$.

This shorthand notation applies to repetitions of blocks as well. For example, in [7, Theorems 3.1 and 4.1], it is shown that the trees $T(4k+3; (0(01)^{2k}00))$ and $T(2k+3; (0(02)^k00))$ are super edge-graceful. The following conjecture is also proposed in [7].

Conjecture. The graphs in the two infinite families $T(2m+3; (02^{2m+1}0))$ and $T(2m+2; (02^{2m}1))$ are super edge-graceful.

In this article, we validate this conjecture by establishing very large lower bounds for the numbers of super edge-graceful labelings of these trees. Denote by $N(G)$ the number of SEG labelings of a graph G. This is positive if and only if G is super edge-graceful.

Theorem 1. $N(T(2m+3, (02^{2m+1}0))) \geq 2^{2m+3} \cdot 3^m \cdot m!$.

Theorem 2. $N(T(2m+2, (02^{2m}1))) \geq 2^{2m+1} \cdot 3^m \cdot m!$.

Before giving the proofs of these theorems, we present some preliminary results.

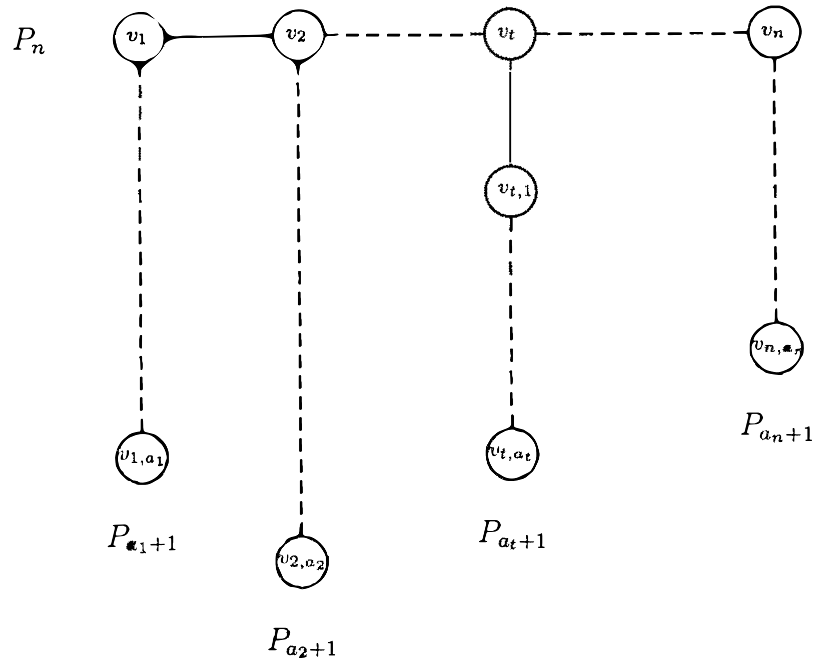

FIGURE 5. The tree $T(n; (a_1, a_2, \ldots, a_n))$

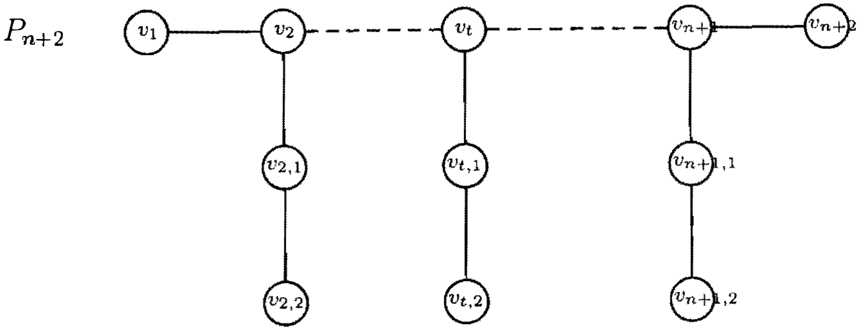

FIGURE 6. The tree $T(n+2; 02^n0)$

3. S-TRIPLET PARTITIONS

We review the notion of S-triplet partition from [7], with slightly different formulation, and restate some results with clearer proofs. A set of

3 distinct positive integers is called an S-triplet if one element is the sum of the remaining two. Let $A \subset \mathbb{N}$. A partition of A is called an S-triplet partition if every part is an S-triplet. We say that such a set is S-triplet partitionable. The following examples play crucial roles in the proofs of our main Theorems 1 and 2.

Proposition 3. *For $t \in \mathbb{N}$, the set $A := \mathbb{N}[1, 6t+1] \setminus \{5t+1\}$ has an S-triplet-partition*

$$A = \bigcup_{i=1}^{t} (\{2i-1, 3t-i+1, 3t+i\} \cup \{2i, 5t-i+1, 5t+i+1\}). \tag{2}$$

Proof. Each 3-element set in (2) is an S-triplet, the third element being the sum of the remaining two. Now we prove that (2) is a partition. For $i = 1, 2, \ldots, t$, let

$$\begin{aligned}(a_{i1}, a_{i2}, a_{i3}) &= (2i-1, 3t-i+1, 3t+i),\\ (a'_{i1}, a'_{i2}, a'_{i3}) &= (2i, 5t-i+1, 5t+i+1).\end{aligned}$$

Note that

$$\begin{aligned}\{a_{i1} : 1 \le i \le t\} &= \mathbb{O}[1, 2t-1] = \mathbb{O}[1, 2t],\\ \{a_{i2} : 1 \le i \le t\} &= \mathbb{N}[2t+1, 3t],\\ \{a_{i3} : 1 \le i \le t\} &= \mathbb{N}[3t+1, 4t],\\ \{a'_{i1} : 1 \le i \le t\} &= \mathbb{E}[2, 2t] = \mathbb{E}[1, 2t],\\ \{a'_{i2} : 1 \le i \le t\} &= \mathbb{N}[4t+1, 5t],\\ \{a'_{i3} : 1 \le i \le t\} &= \mathbb{N}[5t+2, 6t+1].\end{aligned}$$

These sets are pairwise disjoint and

$$\begin{aligned}\{a_{i1} : 1 \le i \le t\} \cup \{a'_{i1} : 1 \le i \le t\} &= \mathbb{O}[1, 2t] \cup \mathbb{E}[1, 2t]\\ &= \mathbb{N}[1, 2t],\\ \{a_{i2} : 1 \le i \le t\} \cup \{a'_{i3} : 1 \le i \le t\} &= \mathbb{N}[2t+1, 3t] \cup \mathbb{N}[3t+1, 4t]\\ &= \mathbb{N}[2t+1, 4t],\\ \{a'_{i2} : 1 \le i \le t\} \cup \{a'_{i3} : 1 \le i \le t\} &= \mathbb{N}[4t+1, 5t] \cup \mathbb{N}[5t+2, 6t+1]\\ &= \mathbb{N}[5t+2, 6t+1].\end{aligned}$$

It is clear that their union is $\mathbb{N}[1, 6t+1] \setminus \{5t+1\}$. ☐

Proposition 4. *For $t \in \mathbb{N}$, the set $B := \mathbb{N}[1, 6t+4] \setminus \{5t+4\}$ has an S-triplet-partition*

$$\begin{aligned}B = &\left(\bigcup_{i=1}^{t} (\{2i-1, 3t-i+3, 3t+i+2\} \cup \{2i, 5t-i+4, 5t+i+4\})\right)\\ &\cup \{2t+1, 2t+2, 4t+3\}.\end{aligned} \tag{3}$$

Proof. Each 3-element set in (3) is an S-triplet, the third element being the sum of the remaining two. Now we prove that (3) is a partition. For $i = 1, 2, \ldots, t$, let

$$\begin{aligned}(b_{i1}, b_{i2}, b_{i3}) &= (2i-1, 3t-i+3, 3t+i+2),\\ (b'_{i1}, b'_{i2}, b'_{i3}) &= (2i, 5t-i+4, 5t+i+4).\end{aligned}$$

Also let

$$(b_{t+1,1}, b_{t+1,2}, b_{t+1,3}) = (2t+1, 2t+2, 4t+3).$$

Then we have

$$\begin{aligned}\{b_{i1} : 1 \le i \le t+1\} &= \mathbb{O}[1, 2t+1],\\ \{b_{i2} : 1 \le i \le t+1\} &= \mathbb{N}[2t+2, 3t+2],\\ \{b_{i3} : 1 \le i \le t+1\} &= \mathbb{N}[3t+3, 4t+3],\\ \{b'_{i1} : 1 \le i \le t\} &= \mathbb{E}[2, 2t] = \mathbb{E}[1, 2t+1],\\ \{b'_{i2} : 1 \le i \le t\} &= \mathbb{N}[4t+4, 5t+3],\\ \{b'_{i3} : 1 \le i \le t\} &= \mathbb{N}[5t+5, 6t+4].\end{aligned}$$

These sets are pairwise disjoint and

$$\begin{aligned}&\{b_{i1} : 1 \le i \le t+1\} \cup \{b'_{i1} : 1 \le i \le t\}\\ &= \mathbb{O}[1, 2t+1] \cup \mathbb{E}[1, 2t+1]\\ &= \mathbb{N}[1, 2t+1],\\ &\{b_{i2} : 1 \le i \le t+1\} \cup \{b_{i3} : 1 \le i \le t+1\}\\ &= \mathbb{N}[2t+2, 3t+2] \cup \mathbb{N}[3t+3, 4t+3]\\ &= \mathbb{N}[2t+2, 4t+3],\\ &\{b'_{i2} : 1 \le i \le t\} \cup \{b'_{i3} : 1 \le i \le t\}\\ &= \mathbb{N}[4t+4, 5t+3] \cup \mathbb{N}[5t+5, 6t+4]\\ &= \mathbb{N}[4t+4, 6t+4] \setminus \{5t+4\}.\end{aligned}$$

The union of these sets is clearly $\mathbb{N}[1, 6t+1] \setminus \{5t+4\}$. ☐

Combining these two propositions, we immediately have

Theorem 5. *For* $m \in \mathbb{N}$, *let*

$$c = \begin{cases} 5t+1, & \text{if } m = 2t, \\ 5t+4, & \text{if } m = 2t+1. \end{cases} \tag{4}$$

The set $C := \mathbb{N}[1, 3m+1] \setminus \{c\}$ *is* S*-triplet-partitionable.*

4. ENUMERATION OF LABELINGS OF TWO BASIC GRAPHS

4.1. **The graph** $T = T(3;(0,2,2))$**.** The graph $T = T(3;(0,2,2))$ is super edge-graceful; see Figure 7 with an SEG labeling. We enumerate all such labelings satisfying (1) without restricting the edge labels to $L(6) = \{\pm 1, \pm 2, \pm 3\}$.

Let $a, b, c \in \mathbb{N}$ satisfy $a + b = c$. We begin with an edge-labeling f of T by $\{\pm a, \pm b, \pm c\}$. If the vertex-labeling f^* induced by (1) takes values in $\{0,\ \pm a,\ \pm b,\ \pm c\}$ and satisfies $f^*(v_3) = 0$, we call the resulting labeling of T a G-labeling from the S-triplet $\{a, b, c\}$.

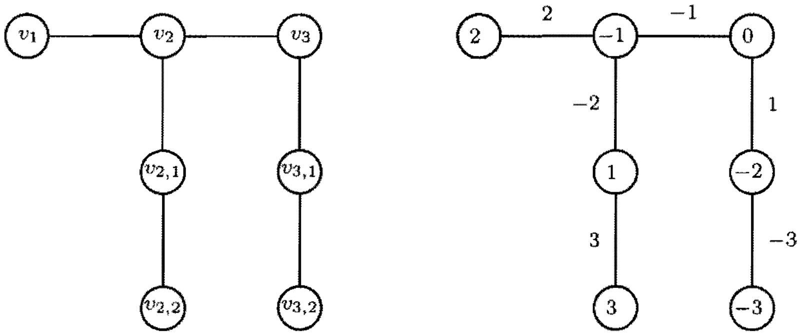

FIGURE 7. The graph T and an SEG labeling

Proposition 6. *Let* $a, b, c \in \mathbb{N}$ *satisfy* $a + b = c$. *The graph* T *has at least* 12 G*-labelings from the* S*-triplet* $\{a, b, c\}$.

Proof. We begin with three distinct G-labelings f_1, f_2, f_3 of T.

Let us define two unary operations P and N on the G-labelings.

(i) $\mathsf{P}f$ is the G-labeling induced by the permutation simultaneously interchanging the values a, b and $-a$, $-b$. Figure 9 shows the effect of P on the labelings f_1, f_2, f_3.

(ii) $\mathsf{N}f$ is the G-labeling induced by the reversal of signs of the values under f. Figure 10 shows the effect of N on the labelings f_1, f_2, f_3.

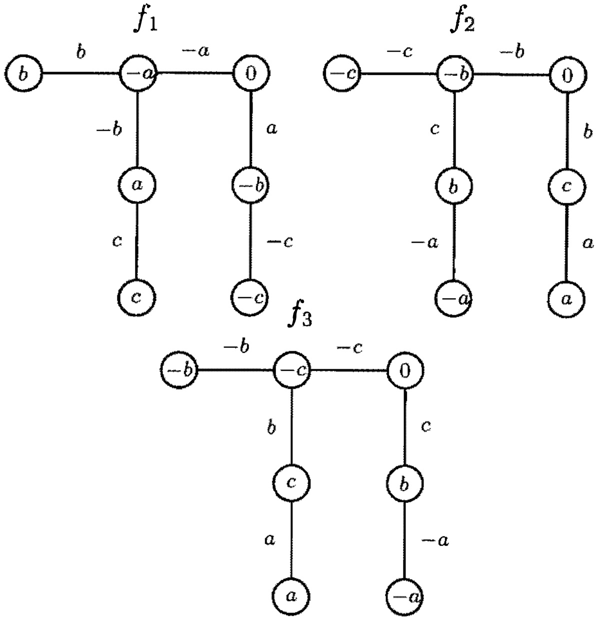

FIGURE 8. Three G-labelings of T

Figure 11 shows the labelings $\mathsf{NP}f_i$, $i = 1, 2, 3$. It is clear that the twelve labelings given in Figures 8, 9, 10, 11 are distinct. This completes the proof of the proposition. ☐

Though Proposition 6 is sufficient for the proof of Theorems 1 and 2, it can indeed be strengthened.

Proposition 7. (1) *If* $a < b < c$ *are natural numbers satisfying* $a + b = c$ *and* $b \neq 2a$, *there are no other* G*-labeling of* T *by the* S*-triplet* $\{a, b, c\}$ *apart from those given in the proof of Proposition 6.*

(2) *If* $a < b < c$ *are natural numbers satisfying* $a + b = c$ *and* $b = 2a$, *there are at least* 4 *more* G*-labelings of* T *by the* S*-triplet* $\{a, b, c\}$ *apart from those given in the proof of Proposition 6.*

Proof. (1) Suppose to the contrary that there is a different G-labeling f, with induced bijection f^*. Since

$$f^*(v_3) = 0 \tag{5}$$

we have

$$f(\{v_{3,1}, v_3\}) = -f(\{v_2, v_3\}). \tag{6}$$

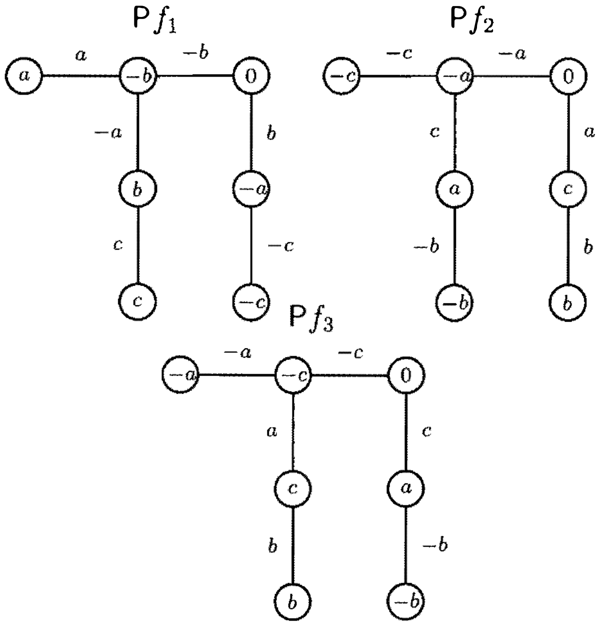

FIGURE 9. Three G-labelings of T from P

Since Pf, Nf and NPf are also G-labelings, we may assume without loss of generality that (6) is realized by

$$f(\{v_{3,1}, v_3\}) = a, \ f(\{v_2, v_3\}) = -a \tag{7}$$

or

$$f(\{v_{3,1}, v_3\}) = c, \ f(\{v_2, v_3\}) = -c. \tag{8}$$

In case of (7), consider the labeling in Figure 12A. Since $y = a + x$ is an equation with $x,\ y \in \{\pm a, \pm b, \pm c\}$, the only possibilities are $(x, y) = (b, c)$, $(-c,\ -b)$. See Figures 13A and 13B.

In Figure 13A, z cannot be any of $\pm a$, b, c. It is either $-b$ or $-c$. Note that $u + z = v$. If $z = -c$, we must have $(u, v) = (a, -b)$ or $(b, -a)$, both impossibilities. It follows that $z = -b$, and from $u + (-b) = v$, we have $(u, v) = (c, a)$, and finally the labeling Pf_2 in Figure 9.

In Figure 13B, z cannot be any of $\pm a$, $-b$, $-c$. It is either b or c. Note that $u + z = v$. If $z = b$, we must have $(u, v) = (a, c)$ or $(-c, -a)$, both impossibilities. It follows that $z = c$, and from $u + c = v$, we have $(u, v) = (-b, a)$, and finally the labeling f_1 in Figure 8.

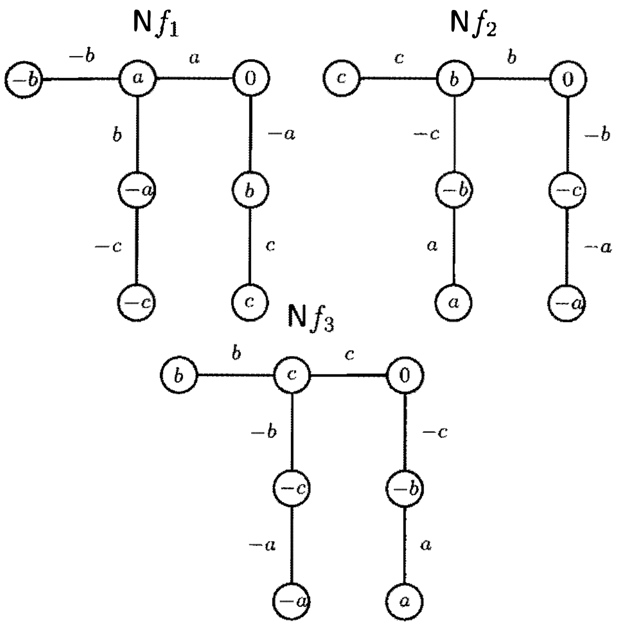

FIGURE 10. Three G-labelings of T from N

In case of (8), consider the labeling in Figure 12B. Since $y = c + x$ is an equation with $x,\ y \in \{\pm a, \pm b, \pm c\}$, the only possibilities are $(x,\ y) = (-a,\ b)$, $(-b,\ a)$. See Figures 14A and 14B.

In Figure 14A, z cannot be any of $-a$, b, $\pm c$. It is either a or $-b$. Note that $u + z = v$. If $z = -b$, we must have $(u, v) = (-a, -c)$ or (c, a), both impossibilities. It follows that $z = a$, and from $u + a = v$, we have $(u, v) = (b, c)$, and finally the labeling f_3 in Figure 8.

In Figure 14B, z cannot be any of a, $-b$, $\pm c$. It is either $-a$ or b. Note that $u + z = v$. If $z = -a$, we must have $(u, v) = (-b, -c)$ or (c, b), both impossibilities. It follows that $z = b$, and from $u + b = v$, we have $(u, v) = (a, c)$, and finally the labeling Pf_3 in Figure 9.

(2) Figures 15 and 16 show one more G-labeling f_4 and the effect of the unary operations P, N, NP on it. These give four more G-labelings distinct from those in Figures 8, 9, 10, and 11. ☐

4.2. **The graph** $T' = T(3; (0, 2, 0))$. Now we consider another super edge-graceful graph $T' = T(3; (0, 2, 0))$:

Again we consider more general labelings of edges by $\{\pm a,\ \pm b\}$, and vertices by $\{0,\ \pm a,\ \pm b\}$. If a bijection $f : E \to \{\pm a, \pm b\}$ induces a

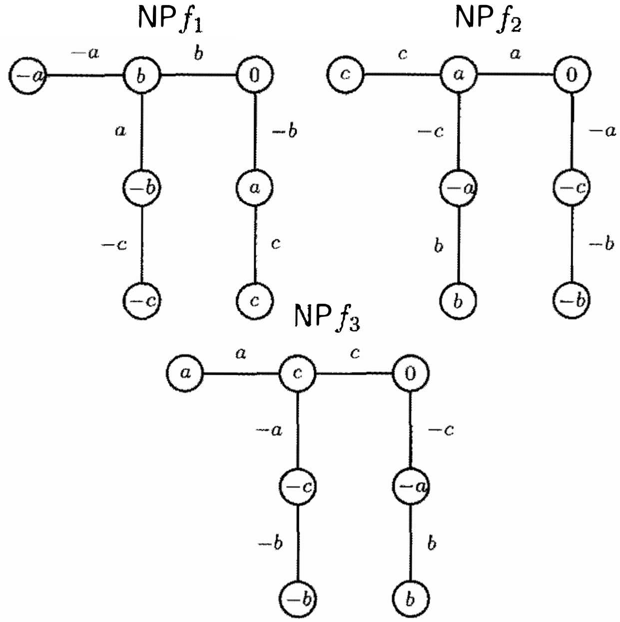

FIGURE 11. Three G-labelings of T from NP

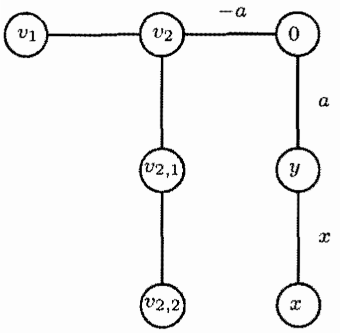

Figure 12A

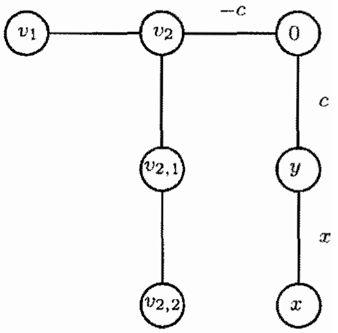

Figure 12B

bijection $f^* : V \to \{0, \pm a, \pm b\}$ satisfying (1), we call it a K-labeling of T' from the pair $\{a, b\}$.

Proposition 8. *Let $a, b \in \mathbb{Z}$ with $a \neq \pm b$. There are at least 8 distinct K-labelings of T' from the pair $\{a, b\}$.*

Proof. We begin with a K-labeling given in Figure 18 below.

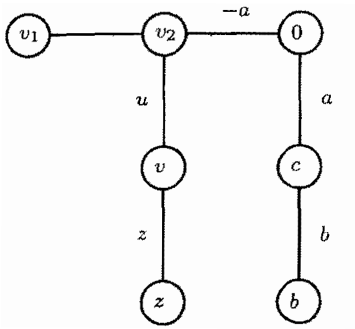

Figure 13A

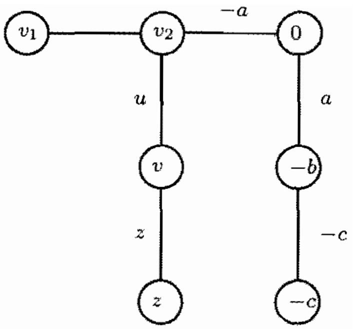

Figure 13B

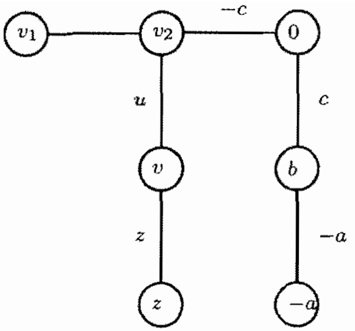

Figure 14A

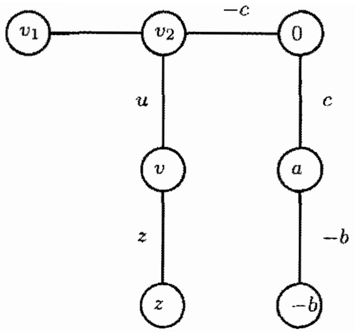

Figure 14B

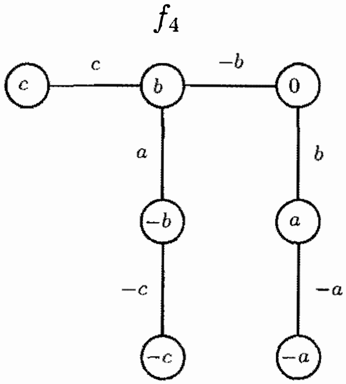

FIGURE 15. A G-labeling f_4 of G in the case $b = 2a$

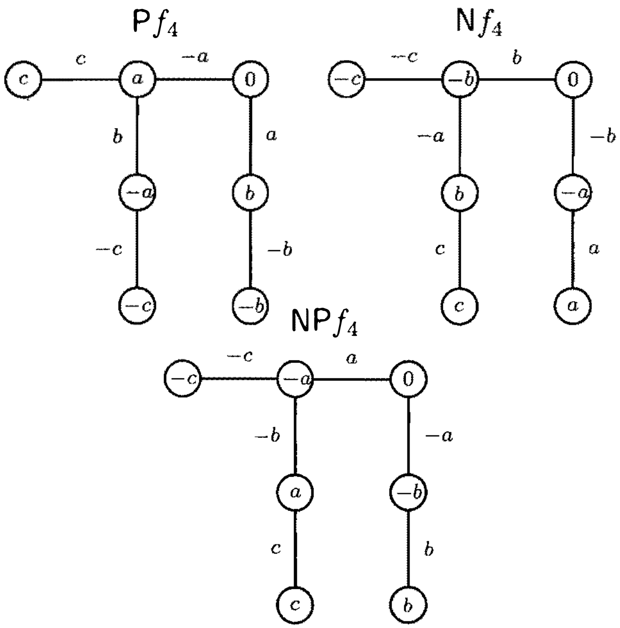

FIGURE 16. Three more G-labelings of T from f_4

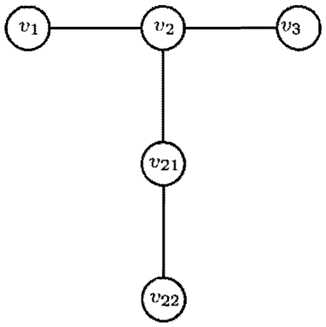

FIGURE 17. The graph $T' = T(3; (0, 2, 0))$

Let us define three unary operations P, A and B on K-labelings.

(i) Pf is the K-labeling induced by the permutation simultaneously interchanging the values a, b and $-a$, $-b$.

(ii) Af is the K-labeling induced by interchanging the values a and $-a$.

(iii) Bf is the K-labeling induced by interchanging the values b and $-b$.

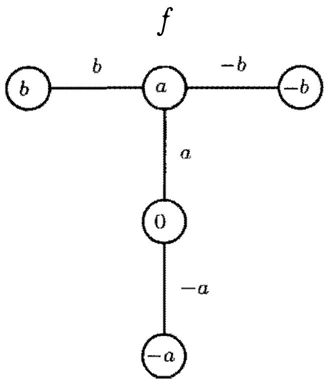

FIGURE 18. A K-labeling of T'

Applying these operations to the labeling f, we have the three in Figure 19.

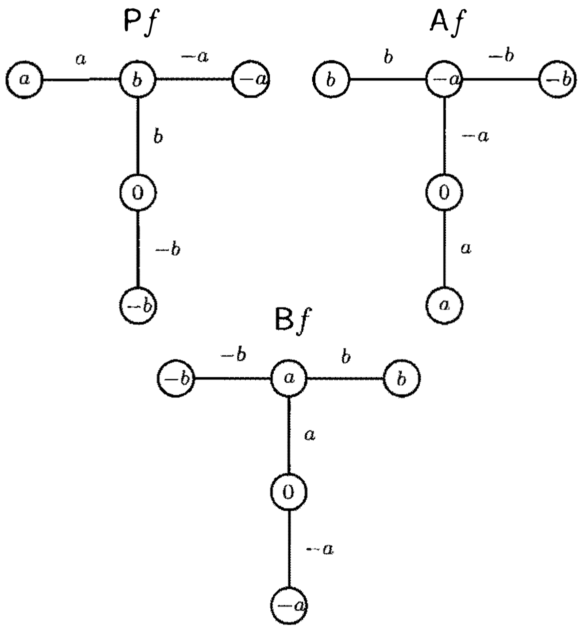

FIGURE 19. Three unary operations on a K-labeling

Figure 20 displays four more labelings APf, BPf, ABf and BAPf. All these are K-labelings distinct from f. This proves the proposition. ☐

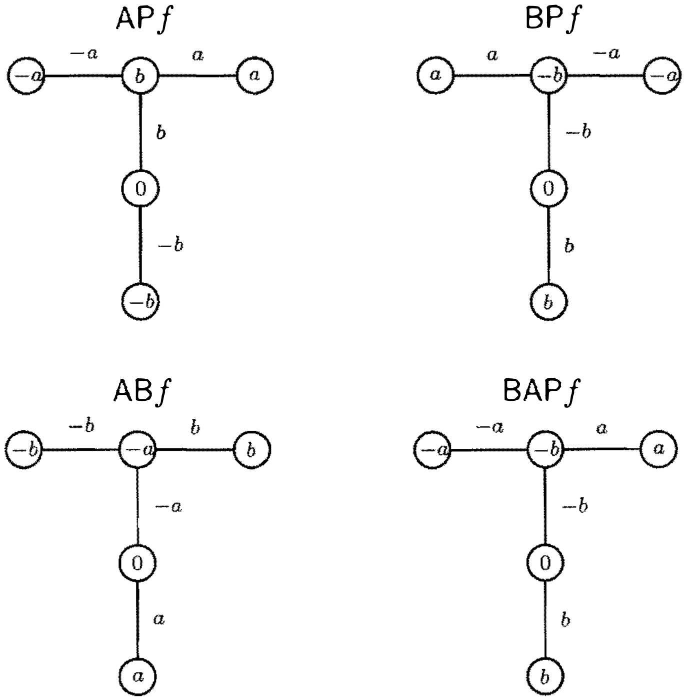

FIGURE 20. Four K-labelings from composite operations

5. PROOF OF THEOREM 1

The tree $T(2m+3, (02^{2m+1}0))$ has $p = 6m+5$ vertices and $q = 6m+4$ edges. It can be regarded as an amalgamation of m copies of T and one copy of T':

Note that $L(q) = \mathbb{Z}[-(3m+2), 3m+2] \setminus \{0\}$. By Theorem 5, the set $C = \mathbb{N}[1, 3m+1] \setminus \{c\}$ is S-triplet-partitionable, where c is given by (4).

Let $\{S_j : 1 \leq j \leq m\}$ be an S-triplet partition of C. We give each copy of T_i a G-labeling from the S-triplet S_j. There are $m!$ possible distributions. For each of these, there are, by Proposition 6, at least 12 G-labelings. For the subgraph T', there are, by Proposition 8, at least 8 K-labelings from $\{c, 3m+2\}$. Therefore, there are at least $12^m \cdot m! \cdot 8 = 2^{2m+3} \cdot 3^m \cdot m!$ labelings of the tree. Since for each K-labeling of T, $f^*(v_3) = 0$, superpositions of the labelings of the components give an SEG labeling of the tree. This completes the proof of Theorem 1.

Figure 22 gives a super edge-graceful labeling of $T(7; (02^5 0))$.

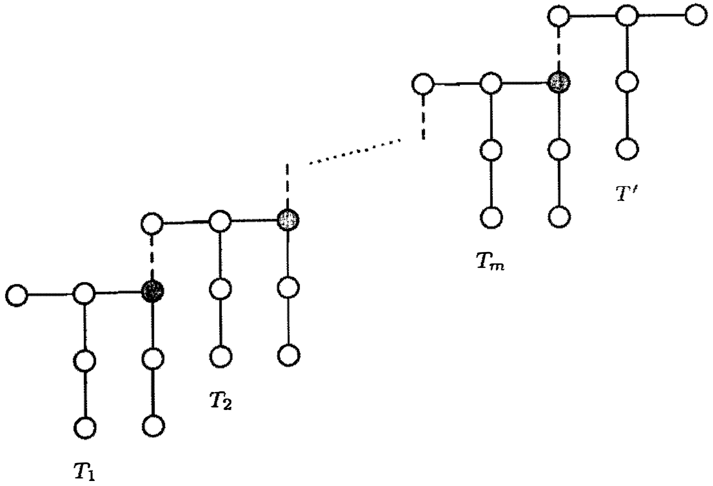

FIGURE 21. The tree $T(2m+3, (02^{2m+1}0))$ as an amalgamation

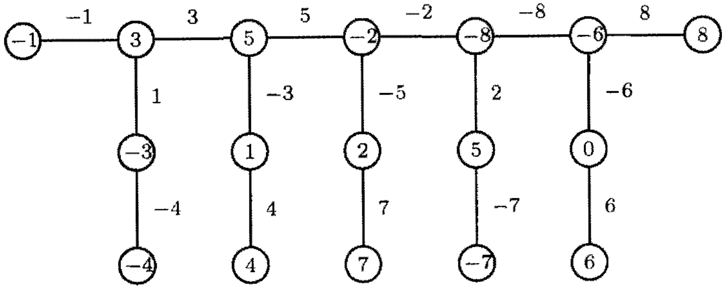

FIGURE 22. A super edge-graceful labeling of $T(7; (02^5 0))$.

6. PROOF OF THEOREM 2

The tree $T(2m+2, (02^{2m}1))$ has $p = 6m+3$ vertices and $q = 6m+2$ edges. It can be regarded as an amalgamation of m copies of T and one copy of P_3. See Figure 23.

In this case, $L(q) = \mathbb{Z}[-(3m+1), 3m+1] \setminus \{0\}$. Consider as above an S-triplet partition $\{S_j : 1 \leq j \leq m\}$ of $C = \mathbb{N}[1, 3m+1] \setminus \{c\}$. Again, there are $m!$ possible distributions of these triplets to the m copies of T, and for each of these, there are at least 12 G-labelings. On the other hand, there are obviously 2 ways of labeling the edges of the P_3 component with $\{c, -c\}$. With the induced vertex labelings, these are shown in Figure 24.

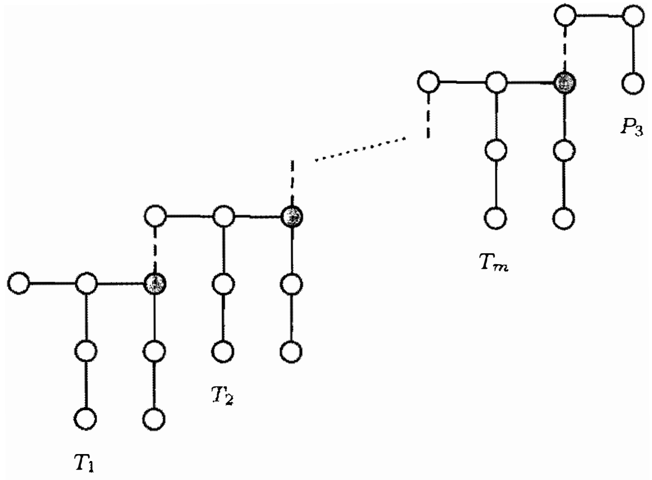

FIGURE 23. The tree $T(2m+3,(02^{2m}1))$ as an amalgamation

FIGURE 24. Labelings of P_3

Therefore, there are at least $12^m \cdot m! \cdot 2 = 2^{2m+1} \cdot 3^m \cdot m!$ labelings of the tree. Since for each K-labeling of T, $f^*(v_3) = 0$, superpositions of the labelings of the components give an SEG labeling of the tree. This completes the proof of Theorem 2.

Figure 25 gives a super edge-graceful labeling of $T(6;(02^4 1))$.

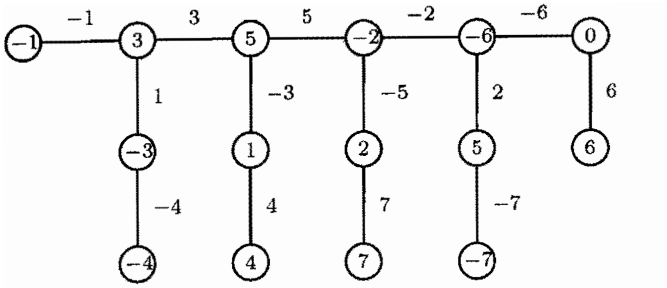

FIGURE 25. A super edge-graceful labeling of $T(6;(02^4 1))$

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