ON SOME EXTREME PROPERTIES OF A CLASS OF SIMPLE GROUPOIDS

Lee Sin-Min and *Sein Aye*

(Received October 18, 1971)

Abstract **We introduce an easy method of constructing a simple groupoid which contains a given groupoid as subgroupoid, and show that the resulting simple groupoid admits only discrete topology and trivial partially ordered relation.**

In [2], J. Ježek has constructed a full embedding of the category of monomorphisms of groupoids into the category of homomorphisms of simple groupoids without idempotents. In a crucial step of his construction, he has shown that every groupoid can be embedded into a simple groupoid (i.e. groupoid which has only two trivial congruences). The simple groupoid constructed by Ježek is quite complicated. Even for a finite groupoid, the resulting simple groupoid contains infinitely many elements.

In [3], we have shown that every groupoid has at least countably ways of embedding into a simple groupoid. The simplest way is given as follows:

Let $\langle G, \circ\rangle$ be a given groupoid. To each $a\in G$, we associate an element $\bar{a}$ and let $\bar{G}=\{\bar{a}\mid a\in G\}$. Assuming that $G\cap\bar{G}=\varnothing$, we set $\tilde{G}=G\cup\bar{G}\cup\{0\}$ where $0\notin\bar{G}\cup G$. Define a multiplication on $\tilde{G}$ as follows:

(1) $0\cdot 0=0$

(2) $\bar{a}\cdot\bar{a}=\bar{\bar{a}}$ for every $\bar{a}\in\bar{G}$.

(3) $0\cdot a=a\cdot 0=\bar{a}$ for every $a\in G$.

(4) $0\cdot\bar{a}=\bar{a}\cdot 0=a$ for every $\bar{a}\in\bar{G}$.

(5) $a\cdot\bar{a}=\bar{a}\cdot a=0$ for every $a\in G$.

(6) $a\cdot b=a\circ b$ for every $a,\ b\in G$.

(7) $\bar{a}\cdot b=b\cdot\bar{a}=\bar{a}\cdot\bar{b}=\bar{b}\cdot\bar{a}=0$ for every $a,\ \bar{b}\in\bar{G},\ \bar{a}\neq\bar{b}$, and for every $b\in G$.

Theorem 1. *$\langle\bar{G}, \circ\rangle$ is a simple groupoid containing G as a subgroupoid.*

Proof. Let θ be a non-trivial congruence of $\tilde{G}$. Then there exist x, $y \in \tilde{G}$, $x \neq y$ such that $x\ \theta\ y$. Consider the following possible cases:

case 1 $x=0$ and $y=a$ for some $a \in G$.

$$\begin{aligned} x\,\theta\,y &\Rightarrow 0\,\theta\,a \\ &\Rightarrow 0\cdot 0\,\theta\,0\cdot a \\ &\Rightarrow 0\,\theta\,\bar{a} \\ &\Rightarrow 0\cdot b\,\theta\,\bar{a}\cdot b \quad \text{for every } b \in G. \\ &\Rightarrow \bar{b}\,\theta\,0 \\ &\Rightarrow 0\cdot\bar{b}\,\theta\,0\cdot 0 \\ &\Rightarrow b\,\theta\,0 \\ &\Rightarrow \theta = \tilde{G}\times\tilde{G}. \end{aligned}$$

case 2 $x=0$ and $y=\bar{a}$ for some $\bar{a} \in \overline{G}$.

By the same argument, $\theta = \tilde{G}\times\overline{G}$.

case 3 $x=a$ and $y=\bar{b}$ for some $a \in G$, $\bar{b} \in \overline{G}$.

$$\begin{aligned} x\,\theta\,y &\Rightarrow a\,\theta\,\bar{b} \\ &\Rightarrow a\cdot\bar{b}\,\theta\,\bar{b}\cdot\bar{b} \\ &\Rightarrow 0\,\theta\,\bar{b} \text{ which reduces to case 2.} \end{aligned}$$

case 4 $x=\bar{a}$ and $y=\bar{b}$ for some $\bar{a}$, $\bar{b} \in \overline{G}$.

$$\begin{aligned} x\,\theta\,y &\Rightarrow \bar{a}\,\theta\,\bar{b} \\ &\Rightarrow \bar{a}\cdot\bar{a}\,\theta\,\bar{a}\cdot\bar{b} \\ &\Rightarrow \bar{a}\,\theta\,0 \text{ which also reduces to case 2.} \end{aligned}$$

case 5 $x=a$ and $y=b$ for some a, $b \in G$.

$$\begin{aligned} x\,\theta\,y &\Rightarrow a\,\theta\,b \\ &\Rightarrow 0\cdot a\,\theta\,0\cdot b \\ &\Rightarrow \bar{a}\,\theta\,\bar{b} \text{ which reduces to case 4.} \end{aligned}$$

Hence $\langle \tilde{G}, \cdot \rangle$ is a simple groupoid which contains G as a subgroupoid.

A topological groupoid $\langle G, \cdot, \tau \rangle$ is a system consisting of a set G on which a binary operation $\cdot$ is defined, and a Hausdorff topology τ such that the operation $\cdot$ is continuous with respect to τ.

Theorem 2. *$\langle \tilde{G}, \cdot \rangle$ admits only discrete topology.*

Proof. Let τ be a Hausdorff topology on $\tilde{G}$ such that the multiplication $\cdot$ is continuous.

Let a be any element of G. We have $0 \cdot \bar{a} = a$. Let W be an open neighbourhood of a which does not contain 0. Then there exist open neighbourhoods U and V such that $0 \in U$, $\bar{a} \in V$ and $U \cdot V \subset W$. By (5) and (7), $0 \in U \subset \{0, \bar{a}\}$, so that $\{0\} = U - \{\bar{a}\}$ is open.

Again, for every $\bar{a} \in \overline{G}$, we have $\bar{a} \cdot \bar{a} = \bar{a}$. Let W be and open neighbourhood of $\bar{a}$ which does not contain 0. Then there exist open neighbourhoods U and V of $\bar{a}$ such that $U \cdot V \subset W$. By (5) and (7), we have $\bar{a} \in U \subseteq \{0, \bar{a}\}$ so that $\{\bar{a}\}$ is open.

Finally, we have $0 \cdot a = \bar{a}$ for every $a \in G$. Take $W = \{\bar{a}\}$ which we have already proved to be open. There exist open neighbourhoods U and V such that $0 \in U$, $a \in V$ and $U \cdot V \subset W$. This implies that $U = \{0\}$, $V = \{a\}$ is that $\{a\}$ is open.

Hence τ is the discrete topology.

We recall that a system $\langle G, \cdot, \leqslant \rangle$ consisting of a groupoid $\langle G, \cdot \rangle$ and a partially ordered relation $\leqslant$ is a partially ordered groupoid if for each $a \in G$ and for $b \leqslant c$ in G, we always have $a \cdot b \leqslant a \cdot c$, $b \cdot a \leqslant c \cdot a$.

Theorem. 3. *$\langle \tilde{G}, \cdot \rangle$ admits only trivial partially ordered relation* $\leqslant = \{x, x \mid x \in \tilde{G}\}$.

Proof. Let $\leqslant$ be a partially ordered relation on $\tilde{G}$. Consider the following possible cases:

case (i) If $0 \leqslant a$ for some $a \in G$, we have

$$0 \cdot \bar{a} \leqslant a \cdot \bar{a}$$
$$\Rightarrow a \leqslant 0$$
$$\Rightarrow 0 = a \text{ which is a contradiction.}$$

case (ii) Similarly, if $0 \leqslant \bar{a}$ for some $\bar{a} \in \overline{G}$, we have $\bar{a} = 0$ which is a contradiction.

case (iii) If $\bar{a} \leqslant \bar{b}$ for some $\bar{a}, \bar{b} \in \overline{G}$, $\bar{a} \neq \bar{b}$, we have

$$\bar{a} \cdot \bar{a} \leqslant \bar{a} \cdot \bar{b}$$
$$\Rightarrow \bar{a} \leqslant 0 \text{ which reduces to case (ii).}$$

case (iv) If $a \leqslant b$ for some $a, b \in G$, $a \neq b$, we obtain

$$0 \cdot a \leqslant 0 \cdot b$$
$$\Rightarrow \bar{a} \leqslant \bar{b} \text{ which reduces to case (iii).}$$

case (v) If $a \leqslant \bar{b}$ for some $a, b \in G$, we have

$$a \cdot \bar{b} \leqslant \bar{b} \cdot \bar{b}$$
$$\Rightarrow 0 \leqslant \bar{b} \text{ which reduces to case (ii).}$$

Hence we conclude that $\leqslant = \{(x, x) \mid x \in \tilde{G}\}$.

Corollary. *Every groupoid can be embedded into a simple groupoid which admits only discrete topology and trivial partially ordered relation.*

REFERENCES

[1] L. Fuchs: Partially ordered algebraic systems. 1963.

[2] J. Ježek: An embedding of groupoids and monomorphisms into simple groupoids. Commentationes Mathematicae Universitatis Carolinae. 11, 1 (1970).

[3] Lee Sin-Min and Sein Aye: On simple groupoids. Nanta Mathematica, Vol. 8, 1975, P. 30—33.

Département de Mathématiques
Université Paris Sud, Centre d'Orsay
91 Orsay, France.

Centre de Mathématiques
Ecole Polytechnique
17 rue Descartes
Paris V° France.