Edge-Graceful and Edge-Magic Labelings of Cartesian Products of Graphs

Karl Schaffer
Dept. of Mathematics
De Anza College
Cupertino, California 95014 U.S.A.

Sin Min Lee
Dept. of Mathematics and Computer Science
San Jose State University
San Jose, California 95192 U.S.A.

ABSTRACT

In this paper we show a relationship between the edge-graceful and the edge-magic conditions of the Cartesian product of edge-graceful graphs. If G is a (p,q) graph in which the edges are labeled 1,2,3,...q so that the induced vertex sums are distinct, mod p, then G is edge-graceful. If such a labeling may be done so that the induced vertex sums are identical, mod p, then G is edge-magic. In 1989 the authors showed that the Cartesian product of even-regular, odd order edge-graceful graphs is also edge-graceful. In this paper we show that in many cases the Cartesian product of these graphs is also edge magic. In particular, the product $C_m X C_n$ of two cycles is always edge-magic. We consider only simple graphs, with no loops or multiple edges.

Key words and phrases: Edge-graceful,edge-magic, Cartesian product, graph
AMS 1991 subject classification: 05C78, O5C25

A graph G=(V,E) with p vertices and q edges is called edge-magic if there is a bijection $f: E \to \{1,2,\ldots,q\}$ such that the induced mapping $f^+ : V \to \mathbf{Z}_p$, given by $f^+(u) = \sum_{v \in N(u)} f(uv) \pmod{n}$ is a constant mapping. It is edge-graceful if f^+ is a bijection. A necessary condition for an edge-magic labeling is:

$$q(q+1) \equiv 0, \pmod{p}, \qquad (1)$$

while a necessary condition for an edge-graceful labeling is (Lo[1]):

$$q(q+1) \equiv \frac{p(p-1)}{2}, \pmod{p} \qquad (2)$$

This latter condition may be more practically stated as $q(q+1) \equiv 0$ or $p/2 \pmod{p}$ depending on whether p is odd or even, respectively.

Another condition is (Lee,Lee, Murthy [2])

$$p \equiv 2 \pmod 4 \text{ implies } G \text{ is not edge graceful.} \qquad (3)$$

One of the authors has conjectured that
Conjecture (Lee[3]): the Lo condition (2) is sufficient for a connected graph to be edge-graceful.

A sub-conjecture of this has also not been proved:
Conjecture (Lee): All odd-order trees are edge-graceful.

Hartsfield and Ringel [4] also call a graph magic if its edges can be labeled with 1,2,...,q such that the vertex sums are identical in Z, and anti-magic if the sums are distinct in Z. Because vertex sums may be identical mod p, but not in Z, magic is thus the stronger concept and implies edge-magic. Edge-graceful is similarly stronger than, and implies, anti-magic.

The authors investigated the edge-gracefulness of cartesian products of connected regular graphs in a paper presented at the Second China-United States Graph Theory Conference in 1989 [5],[6], in which they proved that the product of two odd-order edge graceful graphs is also edge graceful. In particular any product of odd cycles is edge-graceful. S.Wilson and A. Riskin have also presented this result [7] more recently. In this paper we extend that result to edge-magic graphs, using a labeling similar to the edge-graceful case, and present some related results for products of cycles.

The Cartesian product of G and H, G X H, is the graph:
V(G X H) = V(G) X V(H)
E(G X H) = {(u,v),(u',v'), such that either u=u' and (v,v') is in E(H), or v=v' and(u,u') is in E(G)}.

Figure 1 shows that the edge-gracefulness or non edge-gracefulness of G and H do not by themselves determine the edge-gracefulness of G X H. Similarly, figure 2 shows that the edge-magicness of G and H also do not determine the edge-magicness of G X H. (As an example of the connections between these concepts, notice that in the edge-graceful labeling of P_4 X C_3, the subgraph induced by the two central columns of vertices has an induced edge-magic labeling, when 12 is subtracted from each of the edge labels, and the vertex sums are considered in Z_6.)

Edge-gracefulness of products

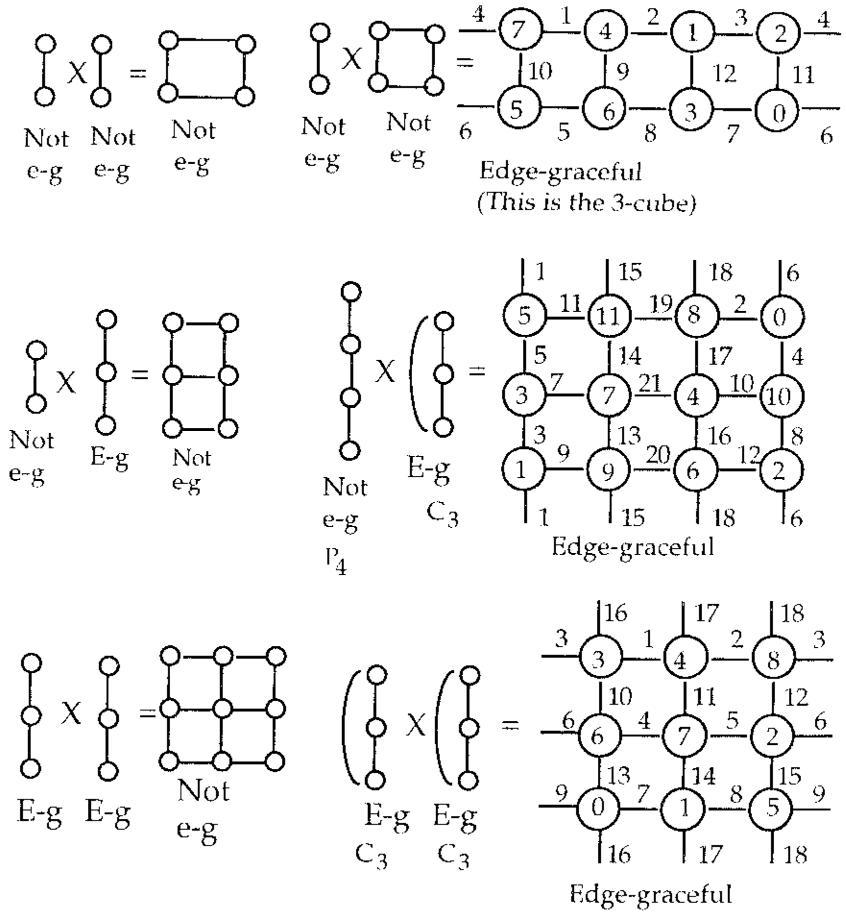


Figure 1

Edge-magicness of products

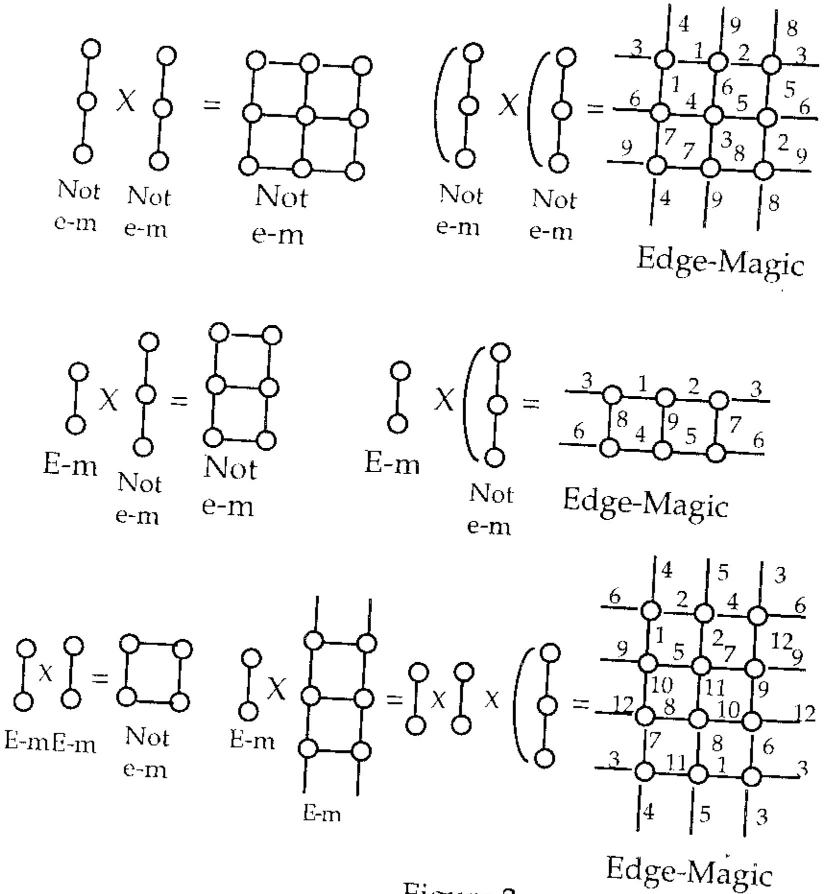
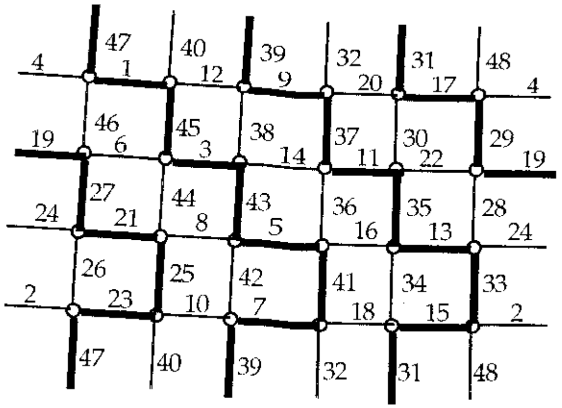


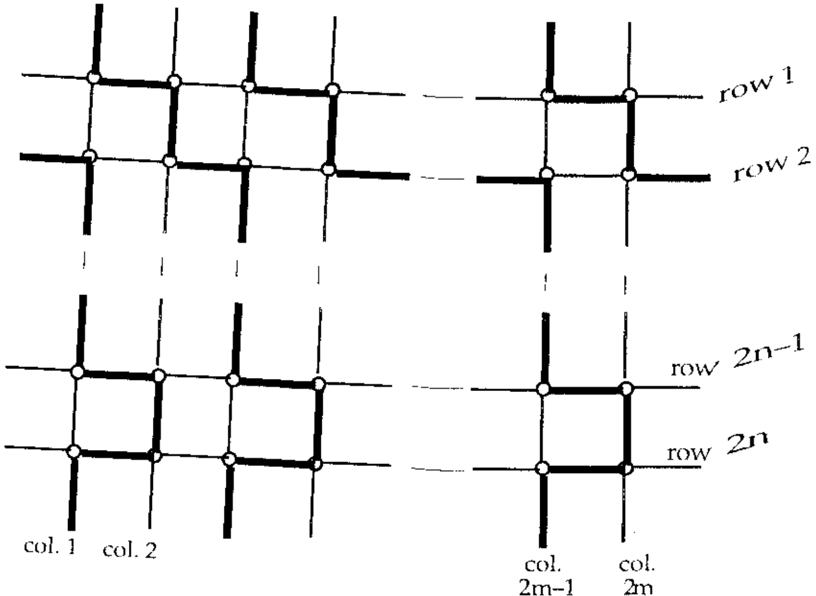
Figure 2

Hartsfield and Ringel [4] demonstrated via a labeling that a bipartite graph which can be decomposed into two Hamiltonian cycles is magic. C_mXC_n is bipartite only if both m and n are even. Figure 3 shows a magic labeling of C_4XC_6 based on the Hartsfield/Ringel theorem. Notice that the sums along the heavy cycle are alternately 48 and 46, while the sums along the lighter cycle alternate between 50 and 52; these sums are paired so that the 52's appear at the same vertices as the 46's, and the 50's are similarly matched with the 48's. Any product of two even cycles may be similarly decomposed into two Hamiltonian cycles, as shown in figure 4, and a similar labeling shows it is magic and also edge-magic.



Magic C_4 X C_6

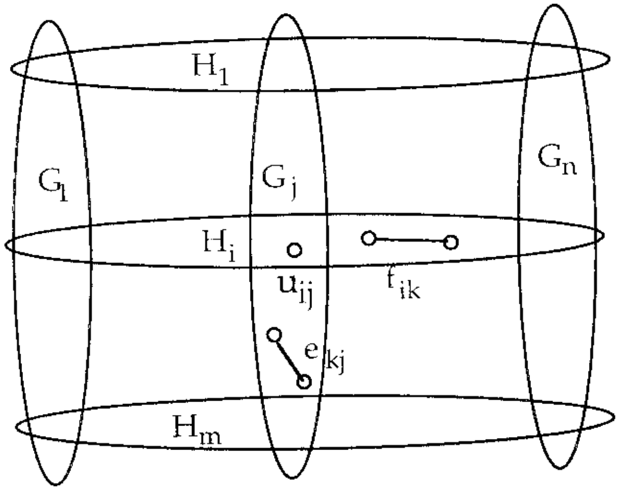
Figure 3



Decomoposition of C_{2m} X C_{2n} into two Hamiltonian cycles

Figure 4

Let G be a c-regular graph with m vertices $u_1, u_2, \ldots, u_m$, r edges $e_1, e_2, \ldots e_r$, and H be a d-regular graph with n vertices $v_1, v_2, \ldots, v_n$, s edges $f_1, f_2, \ldots f_s$. We can consider GXH to be composed of m isomorphic copies of H, namely the subgraphs H_j induced by the vertices (u_j, v_k), as j is held constant and k ranges over1,2,...n, together with n isomorphic copies of G, namely the subgraphs G_j induced by the vertices (u_k, v_j), for k=1,2,...,m. For ease of reference we may refer to the vertices of GXH as $u_{ij}=(u_i,v_j)$. Similarly, the kth edge of H_i will be called f_{ik}, and the kth edge of G_j will be called e_{kj}. (See figure 5.)



G X H

figure 5

We consider mainly edge-graceful graphs which are odd-order and even-regular, and their Cartesian products. We let g and h be edge-graceful labelings of G and H, respectively, with induced mappings g^+ and h^+ respectively of the vertices of G onto $\mathbf{Z}_m$ and of H onto $\mathbf{Z}_n$. We extend the mappings g, h, g^+, and h^+ to the copies G_j and H_i in a manner to be shown below.

When m, the order of G, is odd, and G is c-regular, c must necessarily be even, and the number of edges r must satisfy $r = \frac{c}{2}m$. The edge-graceful labeling of G is therefore accomplished with $\frac{c}{2}$ copies of $\{0,1,2,\ldots, m-1\}$ when labels are considered to be in $\mathbf{Z}_m$. When both G and H are odd-order, even-regular, GXH will be (c+d)-regular, and an edge-graceful or edge-magic labeling of GXH will utilize $\frac{c+d}{2}$ copies of $\{0,1,2,\ldots,mn-1\}$ in $\mathbf{Z}_{mn}$.

The main results are the following related theorems. The second gives a sufficient condition for edge magicness. Although the first has been presented before [5],[7], we give a version of it here to show its connection to the second. In general, the Cartesian product of edge-graceful graphs need not be edge-magic. For example, although K_4 and C_3 are both edge-graceful, $K_4 \times C_3$ does not meet the congruence in condition (1) relating the number of edges, 30, to the number of vertices, 12.

Theorem 1: If G and H are edge-graceful, odd order, regular graphs, then GXH is edge-graceful.

Theorem 2: If G and H are edge-graceful odd-order regular graphs, where G is c-regular and of order m, and H is d-regular and of order n, then GXH is edge-magic if the following condition holds:

(I) gcd(c,mn)=gcd(d,m)=1.

The first requires a lemma:

Lemma: For any integer k, and for n odd, we can find an arrangement $a_0, a_1, \ldots a_{n-1}$ of the n products $\{0k, 1k, 2k, \ldots, (n-1)k\}$ so that the sums $a_i + i$ are distinct, mod n.

Proof of the lemma: Let r=gcd(k,n) and $d=\frac{n}{\gcd(k,n)}$, where d and r must also be odd. The list 0k,1k,2k,...,(n−1)k consists of r copies of each of 0,r,2r,...,(d-1)r, since the products 0k,1k,... begin repeating at dk≡ 0 (mod n). An arrangement which establishes the lemma is $0=a_0=a_1=\ldots=a_{r-1}$, $r=a_r=\ldots=a_{2r-1}, \ldots, (d-1)r=a_{(d-1)r}=\ldots=a_{n-1}$. To see why this arrangement works, note that every integer from 0 to n−1 is expressible uniquely in the form kr+b, where 0 k<d and 0 b<r. Then the sums in question are of the form kr+b+kr. If $k'r+b'+k'r \equiv kr+b+kr$ (mod n), then $2(k-k')r \equiv b'-b$ (mod n). This congruence can only be true if r divides b'−b, since r is a factor of n and of the left side of the congruence; since b and b' are both between 0 and r−1, this is possible only if b'=b. Now $2(k-k')r \equiv 0$ (mod n) implies that k=k', since n, being

odd, is not divisible by 2, and k and k' are both non-negative and less than d. For example, for n=15, k=6, the products 6(0),6(1),...,6(14) are 0,6,12,3,9,0,...9, (mod 15).

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
a_i	0	0	0	3	3	3	6	6	6	9	9	9	12	12	12
$i+a_i$	0	1	2	6	7	8	12	13	14	3	4	5	9	10	11

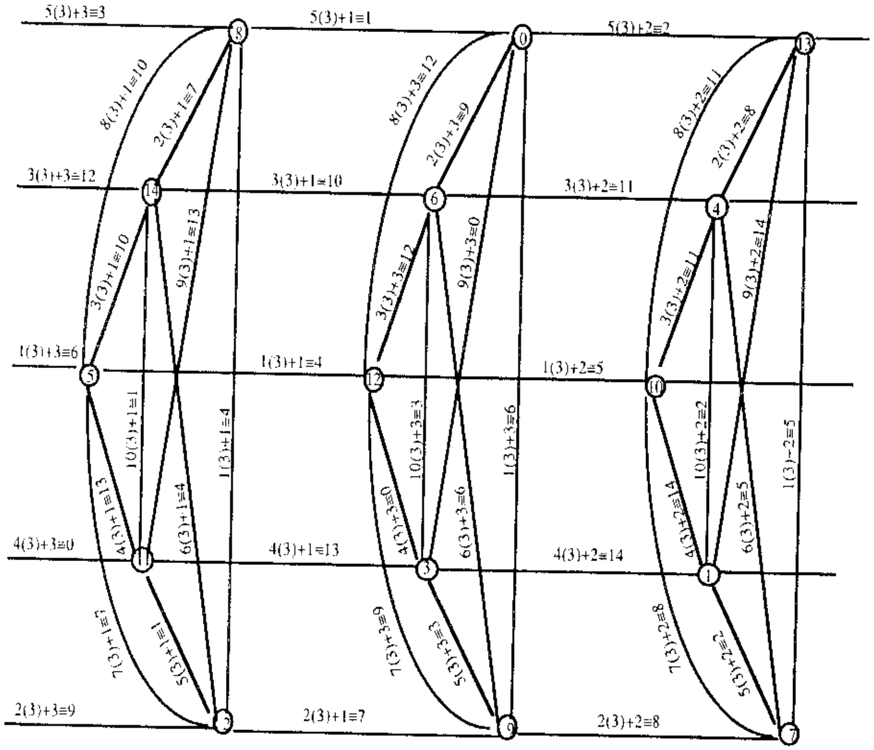
Both theorems may be proved by constructions which begin the same way:
(We will use the examples in figures 6 and 7, of $K_5 X C_3$, to illustrate the argument.)

Step A: Let g and h be edge-graceful labelings of G and H, respectively. Every edge e of GXH will receive a label of the form $n \cdot y(e) + x(e)$, where $x:E(GXH)\rightarrow\{0,1,...,n-1\}$, and $y:E(GXH)\rightarrow\{0,1,...,m-1\}$. The function x will be edge-graceful when restricted to the edges of the H_i, and y will be edge-graceful when restricted to the edges of the G_j. We define x and y by extending the edge graceful labeling h of H to the edges of each H_i, letting $x(f_{ik})=h(f_k)$, where we recall that the edges of H are labeled f_k, and similarly extend g to the edges of G_j by letting $y(e_{kj})=g(e_k)$. The H_i are now each edge-gracefully labeled by x, with respect to mod n, and the function y edge-gracefully labels the edges of each G_j with multiples of n, so that y contributes a distinct multiple of n, with respect to mod mn, to each vertex of G_j. The vertices of GXH thus have distinct sums (so far), mod mn, as each "row" H_i contributes distinct sums, mod n, to each column's vertices, and each "column" G_j contributes distinct multiples of n to each row's vertices. The edges of the H_i do not neccesarily yet include $\frac{d}{2}$ copies of the labels 0,1,...,mn-1.

For example, in figures 6 and 7, each edge label is shown as a sum of two terms, the first a multiple of 3 from 1·3 to 10·3, and the second either 1,2, or 3. At this point the copies of K_5 have received the first term of their edge labels, consisting of multiples of 3, with each multiple of 3 constant "horizontally," that is on the same edge of each K_5 copy. The copies of C_3 have received the second term of their labels, consisting of either 1,2, or 3, with that term constant "vertically," that is on the same edge of each C_3 copy. These portions of the labels are the same in figure 6, the edge-graceful case, as in figure 7, the edge-magic case.

In both the edge-graceful and the edge-magic cases, the function x will be constant on the edges of each G_j, and the function y will be constant on the edges of each H_i; however, different overall assignments will be made to distinguish the edge-graceful case from the edge-magic case.

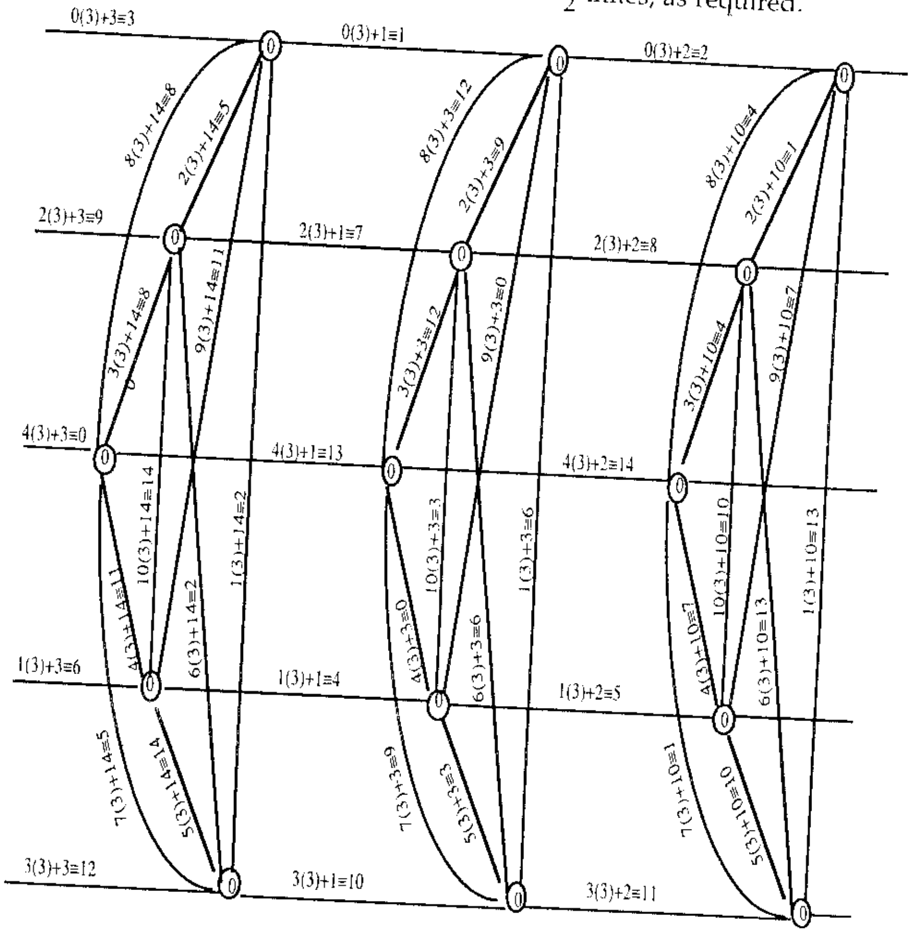
Step B1 (Edge-graceful labeling) To complete the edge-graceful labeling of GXH, define t_j to be the "reduced" element of $\{0,1,...,n-1\}$ that is congruent, mod n, to the sum $h^+(v_j)$. The t_j's are distinct. The function x will be set so that it takes the same value on each edge of a particular G_j, and thus, using the c-regualrity of G, we can apply the lemma to rearrange $0,c,2c,...,(n-1)c$ as $a_0,a_1,...,a_{n-1}$, so that the sums $a_{t_j}+t_j$ are distinct, mod n. For every k, and thus every edge e_{kj} of G_j, let $x(e_{kj})=a_{t_j}$. The mn edges of the collection of all n G_j's now run through the values $\{0,1,...,mn-1\}$ a total of $\frac{c}{2}$ times, mod mn, and the vertex sums are still distinct, mod mn. In fiugre 6 this step supplies the second term for each edge in the copies of K_5. Finally, apply the lemma to rearrange $0,d,2d,...,(m-1)d$ as $b_0,b_1,...,b_{m-1}$, so that the sums b_i+i are distinct mod m. For every edge f_{ik} of H_i let $y(f_{ik})=b_{w_i}$, where as with the t's, w_i is congruent, mod m, to the sum $g^+(u_i)$. In figure 6, this supplies the first term for each edge in the copies of C_3. Up to this point the edge labels of the H_i had run a total of n times through the values $0,1,...,\frac{d}{2}(m)-1$.The mn edges of the collection of H_i's now also run through the values $\{0,1,...,mn-1\}$ a total of $\frac{d}{2}$ times, mod mn, and the vertex sums in GXH are still distinct, mod mn, as well.



Edge-graceful Labeling of K5 X C3

Figure 6

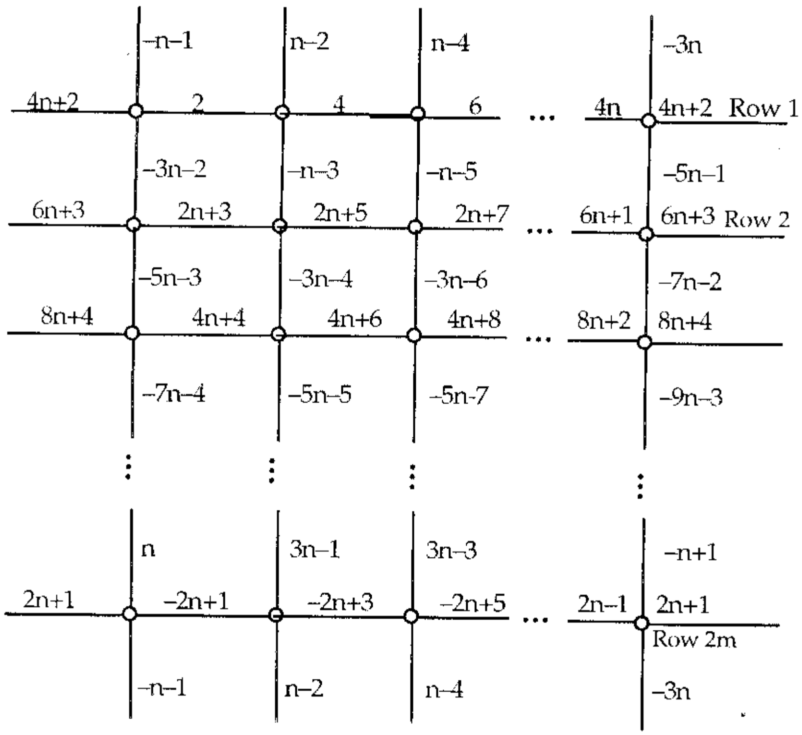
Step B2 (Edge-magic labeling): Completing the edge-magic labeling of GXH after step A, in the cases where condition (I) of theorem 2 is met, is less complex than the edge-gaceful case. For simplicity we choose 0, mod mn, to be the final magic sum at each vertex. Again, the function x will be constant on the edges of the G_j's, and y will be constant on the edges of the H_i's. The congruence $dz+g^+(u_i) \equiv 0$ (mod m) has a unique solution z, as d is relatively prime to m; we assign that value to $y(v_{jk})$ for all k. In figure 7, this supplies the first term, the multiple of 3, for each edge of each C_3 copy. For example, for the vertex at the top left of figure 7, the congruence to be solved is $2z + 20 \equiv 0$ (mod 5), and the solution is $z \equiv 0$, so all edges in the first (top) copy of C_3 receive first term 0(3). The sums contributed by ny are now 0, mod mn. Because c is relatively prime to mn, the congruence equation $cz+h^+(v_j)) \equiv 0$ (mod mn) has a unique solution z; we assign that solution to $x(u_{kj})$, for all k, and the uniqueness, mod n, of the sums $h^+(v_j)$, guarantees that x will not assign the same value to two G_j's. Now the sum at each vertex of GXH is 0, mod mn. Also, the edge assignments within the collection of G_j's run through {0,1,...,mn−1} a total of $\frac{c}{2}$ times, and the assignments within the H_i's run through {0,1,...,mn−1} a total of $\frac{d}{2}$ times, as required.



Edge-magic Labeling of K5 X C3

Figure 7

Corollary: If $m_1, m_2, \ldots, m_l$ are odd, then $C_{m_1} X \ldots X C_{m_l}$ is edge-graceful.
Note that Theorem 2 also shows that many Cartesian products of sets of odd cycles are edge-magic; it may be seen that the smallest such product not covered by this theorem would be $C_3 X (C_{15})^6$.



Edge-magic labeling of $C_{2m} X \ C_{2n+1}$

Figure 8 Magic sum is 1, mod (2m)(2n+1)

Theorem 3: The Cartesian product of two cycles, $C_M X C_N$, is edge-magic.
The cartesian product of two cycles is edge-graceful if and only if they are odd order.
If either M or N is even, the Lo condition precludes edge-gracefulness.
The result may now be shown in three parts:
(i) When M and N are odd, theorems 1 and 2 show how to construct edge-magic and edge-graceful labelings,.
(ii) When M is even and N is odd, the construction in figure 8 shows how to edge-magically label the graph (figure 2 shows the example $C_3 X C_4$). This result will be generalized in theorem 4.
(iii) When both M and N are even, the result follows from the Hartsfield/Ringel result on the decomposition of $C_M X C_N$ into two Hamiltonian cycles, as pointed out earlier.

The authors have also proved, in a similar manner:
Theorem 4: If H is odd-order, d-regular, edge-graceful, and $\gcd(\frac{d}{2}, m)=1$, then $C_{2m} XH$ is edge-magic. (Examples: $C_4 X C_3$, in figure 2, and also figure 8.)

Proof: Since d is even, m must be odd. As in step A, let h be an edge-graceful labeling of H. Label the edge f_{ik} of H_i with $x(f_{ik}) = 2h(f_k) + (i-1)n$, where, as before, f_k is the kth edge of H. Let $x^+(u_{ij})$ be the induced mapping which gives the vertex sum of the edges incident to u_{ij} within H_i. Each H_i layer has $\frac{d}{2}n$ edges, and there are 2m such layers, so the edge labels run 1 through 2mn a total of $\frac{d}{2}$ times within the H_i layers. Note that the edge labels of the H_{2i} are odd, and the edge labels of the H_{2i-1} are even. The d-regularity of H will force each vertex $u_{i+1,j}$ of $H_{i+1}$to have vertex sum dn more than the vertex sum of the corresponding vertex u_{ij} in H_i. We choose the magic sum w as 1 if $\frac{d}{2}$ is odd, 0 if even. For each vertex u_{1j}, label the edge $(u_{2mn,j}, u_{1j})$ by z, and label the edge (u_{1j}, u_{2j}) by $z-\frac{d}{2}n$, where z is a solution of the congruence $z + (z-\frac{d}{2}n) + x^+(u_{1j}) \equiv w \pmod{2mn}$, or

$$2z \equiv w - x^+(u_{1j}) + \frac{d}{2}n \pmod{2mn}$$

The vertex sums $x^+(u_{1j})$ in H_1 are distinct, mod n, because they are twice the numbers 1,2,...,n, and n is odd. Therefore the solutions of these n congruences are distinct, mod 2mn. Note that since x^+ sums an even number of vertex labels, all of which are either even or odd, $x^+(u_{ij})$ must be even; therefore the right hand side of the congruence is even, and has two solutions. We choose either one for z. For the other edges within G_i, label the edge $(u_{ij}, u_{i+1,j})$ with $z - i\frac{d}{2}n$. The total sum at vertex $u_{i+1,j}$ is now

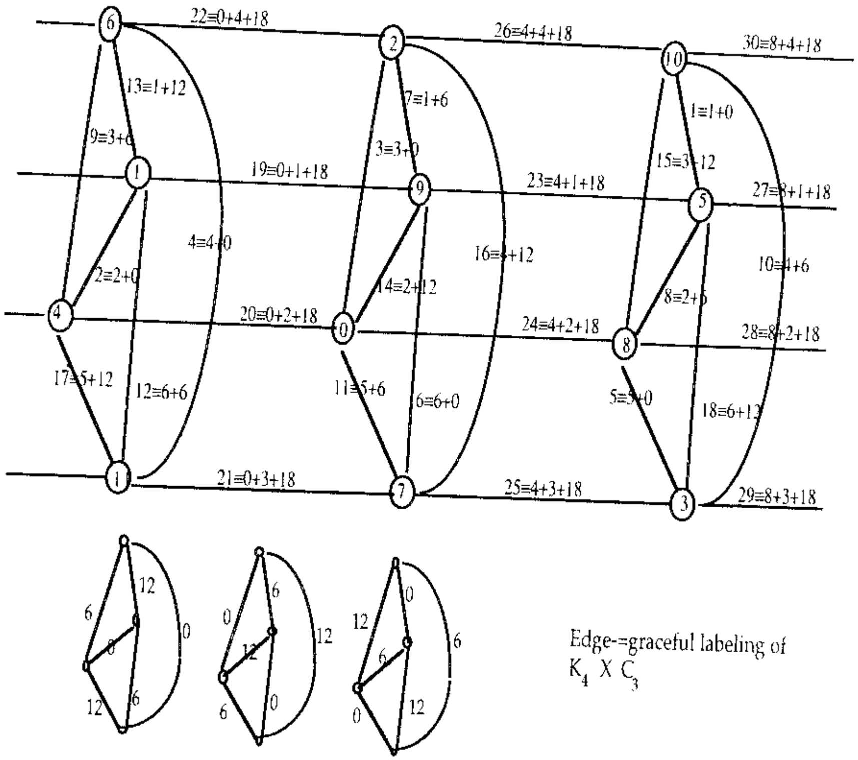
$$(z-i\frac{d}{2}n) + (z-(i+1)\frac{d}{2}n) + x^+(u_{1j}) + idn \equiv 2z - \frac{d}{2}n + x^+(u_{1j}) \equiv w \pmod{2mn}$$

Because $\frac{d}{2}$ is relatively prime to m, the edge labels within one column's copy of C_{2mn} do not repeat.

If H is even-order and d is odd, then the cartesian product of theorem 4 does not satisfy condition (1). The other cases may yet be edge-magic. Certain cartesian products of even-order edge-graceful graphs with cyles may also be edge-gracefully labeled by ad hoc methods. For example,

Theorem 5: Suppose G is odd-regular, edge-graceful, has even order m, m is not divisible by 3, and G may be partitioned into 1-factors. Then G X C_m is edge-graceful. In particular, K_{12k+4} X C_{12k+4} and K_{12k+8} X C_{12k+8} are edge-graceful.

A 1-factor of a graph is a set of edges such that each vertex of the graph is incident with exactly one of the chosen edges. The edges of the 1-factors in the G_i have additional multipes of r added to them, where r is the number of edges in G. See for example the labeling of K_4 X C_3, in figure 9.



Decomposition into 1-factors used to add multipes of 6

Edge-=graceful labeling of K_4 X C_3

Figure 9

Figure 10 summarizes the results on labellings of Cartesian products of regular edge-graceful graphs.

G and H both edge-graceful

<table>
<tr><th colspan="2">GXH: G \ H</th><th rowspan="2">Odd-order, even-regular</th><th colspan="2">Even order</th></tr>
<tr><th></th><th></th><th>odd-regular</th><th>even-regular</th></tr>
<tr><td colspan="2">Odd-order, even-regular</td><td>Edge-graceful
sometimes edge-magic</td><td>sometimes edge-graceful</td><td></td></tr>
<tr><td rowspan="2">Even order</td><td>odd-regular</td><td>sometimes edge-graceful</td><td colspan="2" rowspan="2">Not edge-graceful by Lo condition</td></tr>
<tr><td>even-regular</td><td></td></tr>
</table>

Figure 10

References

[1] S.P. Lo, *On edge-graceful labelings of graphs*, Cong. Num. Vol 50, pp231-241, 1985.

[2] Li Min Lee, Sin Min Lee, and G. MurtyOn Edge-Graceful Labelings of Complete Graphs - Solutions of Lo's Conjecture, *Congressum Numerantum* 62, pp225-233, 1988.

[3] Sin Min Lee and Eric Seah, Edge-Graceful Labelings of Regular Complete k-partite graphs, *Congressum Numerantum* 75, pp 41-50, 1990.

[4] Hartsfield, Nora, and Ringel, Gerhard, *Pearls in Graph Theory*, Academic Press, 1990.

[5] Karl Schaffer and Sin Min Lee, Edge-Graceful Labelings of Cartesian Products, *Second China-United States Graph Theory Conference*, San Francisco, 1989, preprint.

[6] Lee, Sin Min, New Directions in the Theory of Edge-Graceful Graphs, *Proceedings of the 6th Caribbean Conference on Combinatorics & Computing*, 1991, pp 216-231.

[7] Wilson, S., and Riskin, A. *Edge-graceful labellings of odd cycles and their products,*, Bulletin of the ICA, Volume 24, pp 57-64, 1998.

Contributions to the Existence of Some Orthogonal Arrays

D.V. Chopra
Department of Mathematics and Statistics
Wichita State University
Wichita, Kansas 67260-0033, U.S.A.

R. Dios
Department of Mathematical Sciences
New Jersey Institute of Technology
Newark, New Jersey 07102, U.S.A.

Abstract

In this paper we obtain some results on orthogonal arrays (O-arrays) with two symbols and of strength six by deriving some inequalities on the existence of some balanced arrays (B-arrays) of strength six with index set $\underline{\mu}' = (\mu-1,\mu,\mu,\mu,\mu,\mu,\mu-1)$ which we call as Near O-arrays. Consequently we demonstrate that we get better bounds on the number of constraints for some O-arrays when compared to the ones given by Rao (1947).

1. Introduction and Preliminaries.

For ease of reference, we provide here the definition of a balanced array (B-array). A B-array with m rows (constraints), N columns (runs, treatment-combinations), two symbols (say, 0 and 1), and of strength t with index set $\underline{\mu}'=(\mu_0, \mu_1, \ldots \mu_t)$ is an $(m \times N)$ matrix T with elements 0 and 1 such that in every sub-matrix T^* $(t \times N, t \leq m)$ of T, every $(t \times 1)$ vector $\underline{\alpha}$ of weight i (the weight of $\underline{\alpha}$ is the number of i's in it; i=0,1,....,t) appears as a column of T^* exactly μ_i times. The B-array is sometimes denoted by BA $(m, N, 2, t; \mu_0, \mu_1, \ldots \mu_t)$. Clearly $N = \sum_{i=0}^{t} \binom{t}{i} \mu_i$.

The above definition can be easily extended to B-arrays with s (≥ 3) symbols by requiring that every $(t \times 1)$ vector $\underline{\alpha}$ and its permutation $P(\underline{\alpha})$ appear the same number of times in every $(t \times N)$ sub-matrix T^*.

A B-array in which $\mu_i = \mu$ for each i is called an orthogonal array (O-array). Furthermore, there are other combinatorial structures (e.g., balanced incomplete block designs, doubly balanced designs, etc.) closely related to certain