On Balance Index Sets of One-Point Unions of Graphs

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Abstract

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and let $A = \{0, 1\}$. A labeling $f : V(G) \to A$ induces an edge partial labeling $f^* : E(G) \to A$ defined by $f^*(xy) = f(x)$ if and only if $f(x) = f(y)$ for each edge $xy \in E(G)$. For each $i \in A$, let $v_f(i) = |\{v \in V(G) : f(v) = i\}|$ and $e_f(i) = |\{e \in E(G) : f^*(e) = i\}|$. The balance index set of G, denoted $\mathrm{BI}(G)$, is defined as $\{|e_f(0) - e_f(1)| : |v_f(0) - v_f(1)| \le 1\}$. In this paper, exact values of the balance index sets of five new families of one-point union of graphs are obtained, many of them, but not all, form arithmetic progressions.

1 Introduction

A new labeling problem of graphs was considered by Lee, Liu and Tan [6]. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A vertex labeling of G is a mapping f from $V(G)$ into the set $\{0, 1\}$. Corresponding to a vertex labeling f of G, we can define a partial edge labeling f^* of G

in the following way. For each edge $uv \in E(G)$, let

$$f^*(u,v) = \begin{cases} 0 & \text{if } f(u) = f(v) = 0, \\ 1 & \text{if } f(u) = f(v) = 1. \end{cases}$$

Note that if $f(u) \neq f(v)$, then the edge uv is not labeled by f^*. We call f^* the induced partial function from $E(G)$ into the set $\{0,1\}$. Let $v_f(0)$ and $v_f(1)$ denote the number of vertices of G that are labeled 0 and 1 respectively under the mapping f. Likewise, let $e_f(0)$ and $e_f(1)$ denote the number of edges of G that are labeled 0 and 1 respectively under the induced partial function f^*. In other words, for $i = 0, 1$,

$$\begin{aligned} v_f(i) &= |\{u \in V(G) : f(u) = i\}|, \\ e_f(i) &= |\{uv \in E(G) : f^*(uv) = i\}|. \end{aligned}$$

For brevity, when the context is clear, we will drop the subscript and simply write $v(i)$ and $e(i)$. Now we introduce the notion of a balanced graph.

Definition 1.1. A graph G is said to be ***friendly*** if it admits a vertex labeling f such that $|v_f(0) - v_f(1)| \leq 1$.

Definition 1.2. The graph G is called a ***balanced graph*** or said to be ***balanced*** if it admits a vertex labeling f that satisfies the conditions:

$$|v_f(0) - v_f(1)| \leq 1 \qquad \text{and} \qquad |e_f(0) - e_f(1)| \leq 1.$$

Hence, a balanced graph is a friendly graph which has the additional property that $|e_f(0) - e_f(1)| \leq 1$.

Lee, Lee and Ng [6] introduced the following notion in [3] as an extension of their study of balanced graphs.

Definition 1.3. The ***balance index set*** of the graph G is defined as

$$\text{BI}(G) = \{|e_f(0) - e_f(1)| : \text{the vertex labeling } f \text{ is friendly}\}.$$

Example 1. Figure 1 shows a graph G with $\text{BI}(G) = \{0, 1, 2\}$. □

Example 2. For a cycle C_n with vertex set $\{x_1, x_2, \ldots, x_n\}$, we denote by $C_n(t)$ the cycle with a chord x_1x_t. The balance index sets of $C_4(3)$, $C_6(4)$ and $C_6(5)$ are shown in Figure 2. All of them equal to $\{0, 1\}$. □

We note here not every graph has a balance index set consisting of an arithmetic progression.

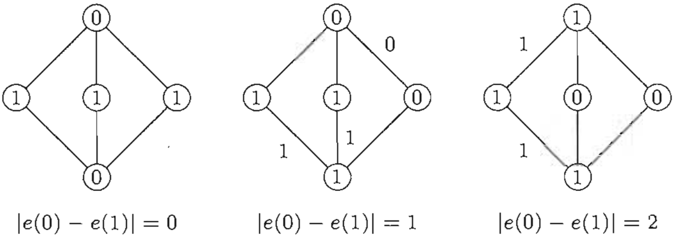

Figure 1: The friendly labelings of a graph G with $\mathrm{BI}(G) = \{0, 1, 2\}$.

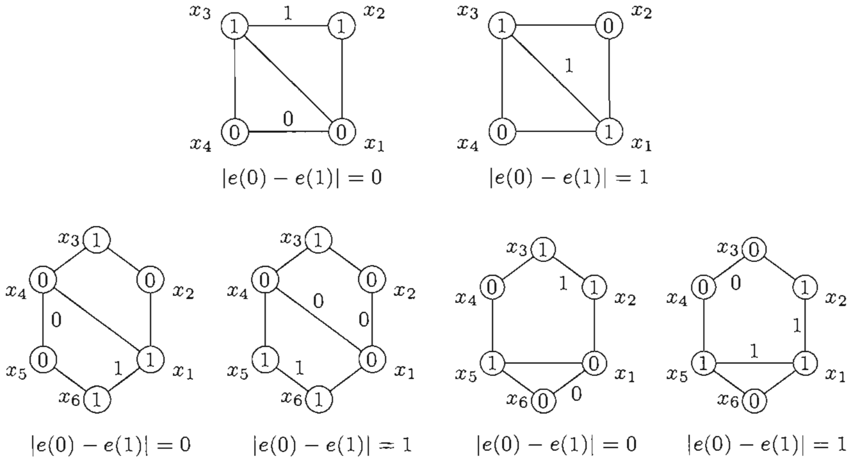

Figure 2: The balance index sets of $C_4(3)$, $C_6(4)$ and $C_6(5)$.

Example 3. The graph $\Phi(1, 3, 1, 1)$ is composed of $C_4(3)$ with an pendant edge appended to each of x_1, x_3 and x_4, and three pendant edges appended to x_2. Figure 3 shows that $\mathrm{BI}(\Phi(1, 3, 1, 1)) = \{0, 1, 2, 3, 4, 6\}$. Note that 5 is missing from the balance index set. □

Some balanced graphs are considered in [2, 3, 6]. In general, it is difficult to determine the balance index set of a graph. The next result is from [7].

Theorem 1.1 *Let $n \geq 4$. Given any t that satisfies $3 \leq t \leq n - 1$,*

$$BI(C_n(t)) = \begin{cases} \{0, 1\} & \text{if } n \text{ is even,} \\ \{0, 1, 2\} & \text{if } n \text{ is odd.} \end{cases}$$

Let (H, x) denote a graph H with a specified vertex x. We construct a new graph $\mathrm{Amal}((H, x), m)$, the amalgamation of m copies of H, by identi-

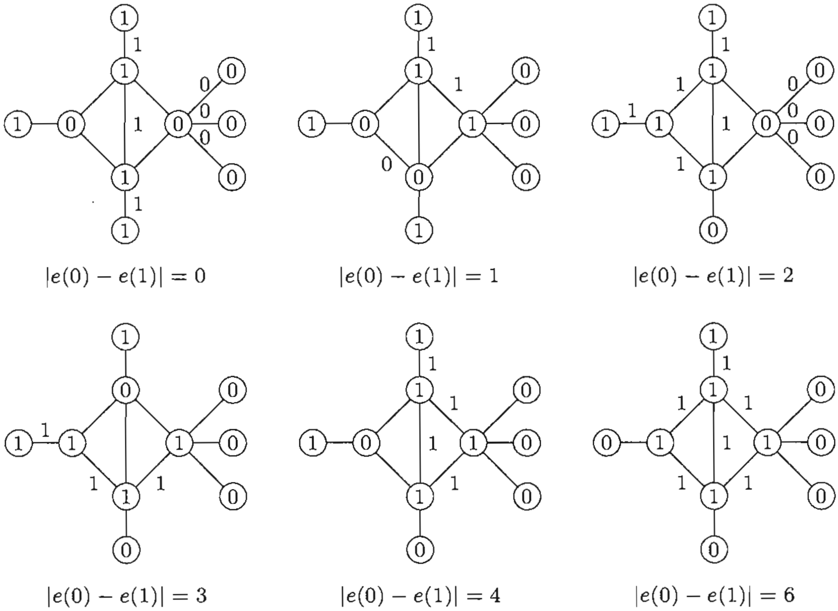

Figure 3: The six friendly labelings of $\Phi(1,3,1,1)$.

fying all the vertices x. The resulting graph is called the ***one-point union*** of (H, x). Shee and Ho [9] used one-point unions to construct numerous cordial graphs. We denote the star with n pendant edges attached to its center by $\mathrm{St}(n)$. Hence the star $\mathrm{St}(n)$ is a tree of diameter two with n pendant vertices. We call its center c and the pendant vertices $x_1, x_2, \ldots, x_n$.

Example 4. If x_1 is a vertex on the path P_2, then $\mathrm{Amal}((P_2, x_1), m)$ is the star $\mathrm{St}(m)$. □

It was shown in [5] that

Theorem 1.2 *For $n \geq 1$,*

$$BI(St(n)) = \begin{cases} \{k\} & \text{if } n = 2k+1, \\ \{k-1, k\} & \text{if } n = 2k. \end{cases}$$

2 Balance Index Sets of Flower Graphs

For a cycle C_n with vertex set $\{x_1, x_2, \ldots, x_n\}$, we will consider x_1 as the specified point.

Definition 2.1. We will call the one-point union $\text{Amal}((C_n, x_1), m)$ a ***flower graph***. For simplicity we will denote it $F(n, m)$.

Example 5. The graph $\text{Amal}((C_3, x_1), m)$ is called a friendship graph. □

The following result can be found in [5]. We provide here a new and shorter proof.

Lemma 2.1 *For any (not necessarily friendly) vertex labeling of C_n, we always have $e(0) - e(1) = v(0) - v(1)$.*

Proof. We may assume, starting from x_1, the first c_1 vertices are labeled 0, the next d_1 vertices 1, the next c_2 vertices 0, the next d_2 vertices 1, and so forth, until we end with c_b 0-vertices and d_b 1-vertices. If all the vertices are labeled the same, define $b = 0$. Then

$$e(0) = \sum_{i=1}^{b}(c_i - 1) = \left(\sum_{i=1}^{b} c_i\right) - b = v(0) - b.$$

Likewise, $e(1) = v(1) - b$, which completes the proof immediately. □

Lemma 2.2 *If a graph contains as a subgraph a cycle of length n which has z vertices labeled 0 and $n - z$ vertices labeled 1, then, restricting to that cycle, $e(0) - e(1) = 2z - n$.*

Proof. The result follows from $e(0) - e(1) = z - (n - z)$. □

Theorem 2.3 *For $n \geq 3$ and $m \geq 1$,*

$$BI(F(n,m)) = \begin{cases} \{m-1\} & \text{if } (n-1)m+1 \text{ is even,} \\ \{m-2, m\} & \text{if } (n-1)m+1 \text{ is odd.} \end{cases}$$

Hence $BI(F(n,m)) = \{m-1\}$ if n is even and m is odd, and $\{m-2, m\}$ otherwise.

Proof. Since changing each vertex label to its complement only changes the sign of $e(0) - e(1)$, we may assume x_1, the center of $F(n, m)$, is labeled 0. Let $v_i(0)$, $v_i(1)$, $e_i(0)$ and $e_i(1)$ denote the respective values in the ith copy of C_n. Then $v(1) = \sum_{i=1}^{m} v_i(1)$, and

$$v(0) = 1 + \sum_{i=1}^{m} (v_i(0) - 1) = 1 - m + \sum_{i=1}^{m} v_i(0).$$

It follows that

$$e(0) - e(1) = \sum_{i=1}^{m}(e_i(0) - e_i(1)) = \sum_{i=1}^{m}(v_i(0) - v_i(1)) = v(0) - v(1) + m - 1.$$

Since the labeling is friendly and $|V(F(n,m))| = (n-1)m+1$, we find

$$v(0) - v(1) = \begin{cases} 0 & \text{if } (n-1)m+1 \text{ is even,} \\ \pm 1 & \text{if } (n-1)m+1 \text{ is odd.} \end{cases}$$

The result follows immediately from $e(0) - e(1) = v(0) - v(1) + m - 1$. □

Example 6. Figure 4 shows the friendly labelings that produce the balance index sets of $F(3,3)$ and $F(3,4)$.

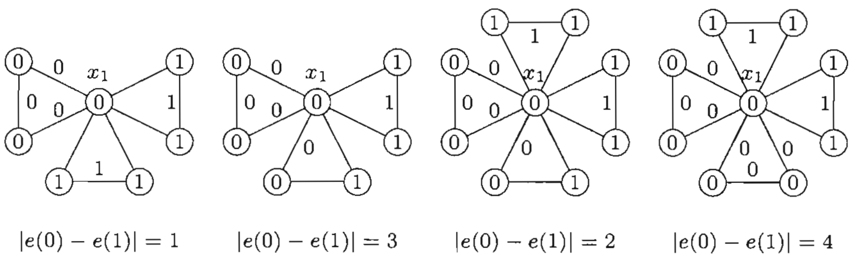

Figure 4: The balance index sets of $F(3,3)$ and $F(3,4)$.

To save space, we could just list the labels of the vertices in each cycle, starting from x_1 to x_n.

C_3	C_3	C_3	C_3	$e(0)$	$e(1)$	$\lvert e(0)-e(1)\rvert$
	000	011	011	3	2	1
	000	001	011	4	1	3
000	001	011	011	4	2	2
000	000	011	011	6	2	4

Using this convention, the friendly labelings of $F(4,m)$, where $2 \le m \le 5$, are summarized in the following table.

C_4	C_4	C_4	C_4	C_4	$e(0)$	$e(1)$	$\lvert e(0)-e(1)\rvert$
			0001	0111	2	2	0
			0000	0111	4	2	2
		0000	0011	0111	5	3	2
	0000	0001	0111	0111	6	4	2
	0000	0000	0111	0111	8	4	4
0000	0000	0011	0111	0111	9	5	4

□

3 Balance Index Sets of One-Point Unions of Certain Trees

Let the vertices on the path P_n be $x_1, x_2, \ldots, x_n$. Then $\mathrm{Amal}((P_n, x_1), m)$ has $(n-1)m+1$ vertices.

Theorem 3.1 *For* $m \geq 2$,

$$BI(Amal((P_2, x_1), m)) = \begin{cases} \{k\} & \text{if } m = 2k+1, \\ \{k-1, k\} & \text{if } m = 2k. \end{cases}$$

and for $n \geq 3$,

$$BI(Amal((P_n, x_1), m)) = \begin{cases} \{0, 1, 2, \ldots, m-1\} & \text{if } (n-1)m+1 \text{ is even}, \\ \{0, 1, 2, \ldots, m\} & \text{if } (n-1)m+1 \text{ is odd}. \end{cases}$$

Proof. Without loss of generality, we may assume x_1 is labeled 0. Using the same argument we used for cycles in the proofs of Lemma 2.1 and Theorem 2.3, we find, restricting to the ith path,

$$e_i(0) - e_i(1) = \begin{cases} v_i(0) - v_i(1) & \text{if } f(x_n) = 1, \\ v_i(0) - v_i(1) - 1 & \text{if } f(x_n) = 0. \end{cases}$$

Since $\sum_{i=1}^{m} v_i(0) = v(0) + m - 1$, we find, over the entire $\mathrm{Amal}((P_n, x_1), m)$,

$$e(0) - e(1) = \sum_{i=1}^{m} v_i(0) - \sum_{i=1}^{m} v_i(1) - m^* = v(0) - v(1) + m - m^* - 1,$$

where m^* denotes the number of paths with $f(x_n) = 0$.

First we consider the special case of $n = 2$. If $m = 2k+1$, then $v(0) = v(1)$, and we must have $m^* = k$; hence $e(0) - e(1) = k$. If $m = 2k$, we could have $m^* = k$ or $m^* = k-1$. If $m^* = k$, we have $v(0) - v(1) = 1$, hence $e(0) - e(1) = k$. If $m^* = k-1$, we have $v(0) - v(1) = -1$, in which case $e(0) - e(1) = k - 1$. This establishes the case of $n = 2$.

Next, consider $n \geq 3$. Since $0 \leq m^* \leq m$, we find

$$e(0) - e(1) \in \begin{cases} \{-1, 0, 1, \ldots, m-1\} & \text{if } v(0) - v(1) = 0, \\ \{0, 1, 2, \ldots, m\} & \text{if } v(0) - v(1) = 1, \\ \{-2, -1, 0, \ldots, m-2\} & \text{if } v(0) - v(1) = -1. \end{cases}$$

The proof will be completed if we can show that there always exists a friendly labeling with $v(0) - v(1) \in \{0, 1\}$ and any m^* between 0 and m, inclusive.

Pick any m^* paths, and set $f(x_n) = 0$, and assign $f(x_n) = 1$ on the other $m - m^*$ paths. Setting $f(x_{n-1}) = 1 - f(x_n)$ ensures that there are equal number of 0- and 1-vertices among the x_{n-1}'s and x_n's. Now we need to label $x_2, x_3, \ldots, x_{n-2}$ in each path. Label them 0 in the odd-numbered paths (except the last when m is odd), and 1 in the even-numbered path. If m is odd, we still have to label $x_2, x_3, \ldots, x_{n-2}$ in the last path. Label $\lfloor (n-3)/2 \rfloor$ of them with 0, and the remaining $\lceil (n-3)/2 \rceil$ vertices with 1. The result is a friendly labeling with $v(0) - v(1) \in \{0, 1\}$ and m^* paths end with $f(x_n) = 0$. □

Example 7. The friendly labelings of Amal$((P_7, x_1), 3)$ that produce its balance index set are listed below. Each row in the table displays, separately, the vertex labeling of the three paths of length 7, starting from x_1 to x_7.

P_7	P_7	P_7	$e(0)$	$e(1)$	$\|e(0) - e(1)\|$
0000001	0111101	0001101	7	4	3
0000010	0111101	0001101	6	4	2
0000010	0111110	0001101	6	5	1
0000010	0111110	0001110	6	6	0

The friendly labelings of Amal$((P_6, x_1), 5)$ are described below.

P_6	P_6	P_6	P_6	P_6	$e(0)$	$e(1)$	$\|e(0) - e(1)\|$
000001	011101	000001	011101	001101	9	5	4
000010	011101	000001	011101	001101	8	5	3
000010	011110	000001	011101	001101	8	6	2
000010	011110	000010	011101	001101	7	6	1
000010	011110	000010	011110	001101	7	7	0
000010	011110	000010	011110	001110	7	8	1

Note that in the last case, $m = m^*$, hence $e(0) - e(1) = -1$. □

When $m = 1$, we could have $e(0) - e(1) = -2$, provided $v(0) - v(1) = -1$ and $m^* = 1$. This is not possible when $n = 3$, but is always possible for any odd $n \geq 5$, because we can label the vertices $0101 \cdots 01110$. We obtain the following result.

Corollary 3.2 *For $n \geq 2$,*

$$BI(P_n) = \begin{cases} \{0\} & \text{if } n = 2, \\ \{0, 1\} & \text{if } n = 3 \text{ or } n \geq 4 \text{ is even}, \\ \{0, 1, 2\} & \text{if } n \geq 5 \text{ is odd}. \end{cases}$$

The one-point union Amal$((\mathrm{St}(n), x_1), m)$ is a tree rooted at x_1, which has m children, each of which has $n-1$ children. Thus Amal$((\mathrm{St}(n), x_1), m)$ has $nm + 1$ vertices.

Theorem 3.3 *For $m, n \geq 1$, define*

$$\begin{aligned} T_1 &= \left\{ \left| (n-1)i - \frac{(n-2)m+1}{2} \right| : 0 \leq i \leq m \right\}, \\ T_2 &= \left\{ \left| (n-1)i - \frac{(n-2)m}{2} \right| : 0 \leq i \leq m \right\}, \\ T_3 &= \left\{ \left| (n-1)i - \frac{(n-2)m}{2} - 1 \right| : 0 \leq i \leq m \right\}, \end{aligned}$$

then

$$BI(Amal((St(n), x_1), m)) = \begin{cases} T_1 & \text{if } nm+1 \text{ is even,} \\ T_2 \cup T_3 & \text{if } nm+1 \text{ is odd.} \end{cases}$$

Proof. Without loss of generality, we may assume x_1 is labeled 0. Assume x_1 is adjacent to i vertices, which are the centers of the stars $\mathrm{St}(n)$, that are labeled 0 and $m-i$ other centers that are labeled 1. Further assume that among the $(n-1)i$ pendant vertices adjacent to these 0-vertices, j are labeled 0, the other $(n-1)i-j$ labeled 1. In a similar manner, among the $(n-1)(m-i)$ vertices adjacent to the 1-vertices in the neighborhood of x_1, assume k of them are labeled 0, and the other $(n-1)(m-i)-k$ labeled 1. Then

$$\begin{aligned} e(0)-e(1) &= i+j-(n-1)(m-i)+k \\ &= (n-1)i+i+j+k-(n-1)m. \end{aligned}$$

From $v(0) = 1+i+j+k$ and $v(1) = nm+1-v(0)$, we deduce that $v(0)-v(1) = 1+2(i+j+k)-nm$. If $nm+1$ is even, then $v(0) = v(1)$, and $i+j+k = \frac{nm-1}{2}$, thus

$$\begin{aligned} e(0)-e(1) &= (n-1)i + \frac{nm-1}{2} - (n-1)m \\ &= (n-1)i - \frac{(n-2)m+1}{2}. \end{aligned}$$

If $nm+1$ is odd, then $v(0)-v(1) = \pm 1$, which leads to

$$e(0)-e(1) = \begin{cases} (n-1)i - \frac{(n-2)m}{2} & \text{if } v(0)-v(1) = 1, \\ (n-1)i - \frac{(n-2)m}{2} - 1 & \text{if } v(0)-v(1) = -1. \end{cases}$$

Since their values depend on i only, all of them are attainable. The result follows from $0 \leq i \leq m$. □

Remark. An effective way to compute $\mathrm{BI}(\mathrm{Amal}((\mathrm{St}(n), x_1), m))$ is to first compile a list (or two) of values of $e(0)-e(1)$. If $nm+1$ is even, the list starts with $-\frac{(n-2)m+1}{2}$, increments by $n-1$, and ends at $\frac{nm-1}{2}$. If $nm+1$ is odd, we need two lists. The first starts with $-\frac{(n-2)m}{2}$, increments by $n-1$, and ends at $\frac{nm}{2}$. The second list can be obtained from the first by subtracting 1 from each value. The last step is to take absolute values to form the balance index set.

Example 8. When $m = 1$, $\mathrm{Amal}((\mathrm{St}(n), x_1), m) = \mathrm{St}(n)$. If $n+1$ is even, the list is $\{-\frac{n-1}{2}, \frac{n-1}{2}\}$. If $n+1$ is odd, we have $\{-\frac{n-2}{2}, \frac{n}{2}\} \cup \{-\frac{n}{2}, \frac{n-2}{2}\}$. Therefore $\mathrm{BI}(\mathrm{St}(n)) = \{\frac{n-1}{2}\}$ if n is odd, and $\{\frac{n}{2}-1, \frac{n}{2}\}$ if n is even; this is precisely what Theorem 1.1 asserts. □

Example 9. When $n = 1$, $\mathrm{Amal}((\mathrm{St}(n), x_1), m)$ reduces to $\mathrm{St}(m)$. When m is odd, we have $m+1$ even, and the balance index set is $\{\frac{m-1}{2}\}$. If m is even, we need two lists because $m+1$ is odd. In this case, the balance index set is $\{\frac{m}{2}\} \cup \{\frac{m}{2}-1\} = \{\frac{m}{2}-1, \frac{m}{2}\}$. The results again agree with Theorem 1.1. □

Example 10. Note that $\mathrm{Amal}((\mathrm{St}(2), x_1), m) = \mathrm{Amal}((P_3, x_1), m)$. Since $2m+1$ is always odd, we need two lists of values of $e(0) - e(1)$, and the values are incremented by 1, hence they are consecutive integers. The first list covers 0 through m, and the second -1 through $m-1$. Therefore $\mathrm{BI}(\mathrm{Amal}((P_3, x_1), m)) = \{0, 1, 2, \ldots, m\}$. This agrees with Theorem 3.1. □

Corollary 3.4 *Let $G = Amal((St(3), x_1), m)$, where $m \geq 1$. Then*

$$BI(G) = \begin{cases} \{1, 3, 5, \ldots, (3m-1)/2\} & \text{if } m \equiv 1 \pmod 4, \\ \{0, 2, 4, 6, \ldots, (3m-1)/2\} & \text{if } m \equiv 3 \pmod 4, \\ \{0, 1, 2, 3, \ldots, 3m/2\} & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

Proof. Since the increment is 2, each list mentioned in the Remark above consists entirely of odd numbers or even numbers. If m is odd, $3m+1$ is even, the lists starts from $-\frac{m+1}{2}$, and ends with $\frac{3m-1}{2}$. Hence the numbers are odd if $m \equiv 1 \pmod 4$, and even if $m \equiv 3 \pmod 4$. Since $\frac{m+1}{2} \leq \frac{3m-1}{2}$, it suffices to consider the nonnegative values, which give the fist two results listed above. If m is even, we need to compile two lists, one containing odd numbers and the other even numbers. This gives the last result. □

Example 11. Recall that $\mathrm{Amal}((\mathrm{St}(3), x_1), m)$ is a tree rooted at x_1, which is adjacent to the centers c of the stars, and each c in turn has x_2 and x_3 as its children. The vertices c, x_2 and x_3 form a subtree T. To save space, we display the labeling of each copy of T in the form of $f(x_2)$-$f(c)$-$f(x_3)$. The friendly labelings of $\mathrm{Amal}((\mathrm{St}(3), x_1), 2)$ are displayed below in this manner. See Figure 5.

T	T	$e(0)$	$e(1)$	$\|e(0)-e(1)\|$
0-0-1	1-1-1	2	2	0
0-0-0	1-1-1	3	2	1
1-0-1	1-0-1	2	0	2
0-0-1	1-0-1	3	0	3

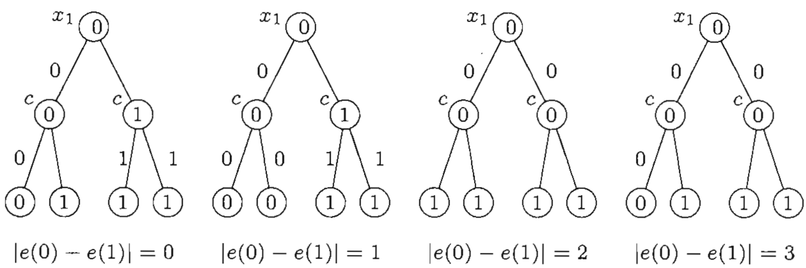

Figure 5: The balance index set of $\text{Amal}((\text{St}(3), x_1), 2)$.

The following table depicts the friendly labelings of $\text{Amal}((\text{St}(3), x_1), 3)$ that produce its balance index set.

T	T	T	$e(0)$	$e(1)$	$\|e(0)-e(1)\|$
0-0-0	0-1-1	1-1-1	3	3	0
0-0-0	1-0-1	1-1-1	4	2	2
0-0-1	1-0-1	1-0-1	4	0	4

□

When $n \geq 4$, the nice pattern we have seen thus far starts to break apart. In fact, the balance index set may not even contain an arithmetic progression. Here is the reason. After taking absolute value of $e(0) - e(1)$, the balance index set is in effect the union of two or four, depending on whether $nm + 1$ is even or odd, sets of numbers. Although each set by itself consists of an arithmetic progression, their union needs not form an arithmetic progression. For example, when $n = 7$ and $m = 5$, we find

$$\{e_f(0) - e_f(1) : f \text{ is friendly}\} = \{-13, -7, -1, 5, 11, 17\}.$$

Hence

$$\text{BI}(\text{Amal}((\text{St}(7), x_1), 5)) = \{1, 7, 13\} \cup \{5, 11, 17\} = \{1, 5, 7, 11, 13, 17\}.$$

In addition, because the sets are of different cardinalities, some entries in what appears to be an arithmetic progression will be missing. For instance, when $n = 4$ and $m = 5$, we find

$$\{e_f(0) - e_f(1) : f \text{ is friendly}\} = \{-5, -2, 1, 4, 7, 10\} \cup \{-6, -3, 0, 3, 6, 9\}.$$

Therefore

$$\begin{aligned}\text{BI}(\text{Amal}((\text{St}(4),x_1),5)) &= \{2,5\}\cup\{1,4,7,10\}\cup\{3,6\}\cup\{0,3,6,9\}\\ &= \{0,1,2,\ldots,10\}-\{8\}.\end{aligned}$$

Example 12. Sometimes the missing entries in the balance index set form an arithmetic progression, as in

$$\text{BI}(\text{Amal}((\text{St}(4),x_1),4)) = \{1,2,3,\ldots,8\}-\{3,6\}.$$

Other times, the missing entries can be split into several arithmetic progressions:

$$\text{BI}(\text{Amal}((\text{St}(6),x_1),5)) = \{0,1,2,\ldots,15\}-(\{2,7,12\}\cup\{3,8,13\}).$$

But in general, the missing entries may not fit any pattern:

$$\begin{aligned}\text{BI}(\text{Amal}((\text{St}(5),x_1),7)) &= \{1,3,5,\ldots,17\}-\{15\},\\ \text{BI}(\text{Amal}((\text{St}(6),x_1),7)) &= \{0,1,2,\ldots,21\}-\{2,3,7,8,12,13,17,18,19\},\\ \text{BI}(\text{Amal}((\text{St}(6),x_1),8)) &= \{1,2,3,\ldots,24\}-\{5,10,15,20,21,22\}.\end{aligned}$$

It would be a challenging project to find the exact values of these balance index sets. □

4 Balance Index Sets of One-Point Unions of Some Unicyclic Graphs

Let $U(n)$ be the graph consisting of an edge appended to any vertex of C_n. Let the vertices on the cycle be $x_1, x_2, \ldots, x_n$. Assume the pendant edge joins x_1 to the pendant vertex c. Note that $\text{Amal}((U(n),c),m)$ has $nm+1$ vertices.

Theorem 4.1 *Let $G = Amal((U(n),c),m)$, where $m \geq 1$. Then*

$$BI(G) = \begin{cases} \{0,1,2,\ldots,\max(1,m-1)\} & \text{if } nm+1 \text{ even,}\\ \{0,1,2,\ldots,\max(2,m)\} & \text{if } nm+1 \text{ is odd.}\end{cases}$$

Proof. Without loss of generality, we may assume c is labeled 0. Let m^* be the number of 0-vertices it is adjacent to. Let $v_i(0)$ be the number of 0-vertices on the ith copy of C_n. Lemma 2.2 asserts that, restricting to the ith copy of C_n, $e(0)-e(1) = 2v_i(0)-n$. Therefore, over $\text{Amal}((U(n),c),m)$,

$$e(0)-e(1) = m^* + 2\sum_{i=1}^{m} v_i(0) - nm = m^* + 2v(0) - nm - 2.$$

This gives

$$e(0) - e(1) = \begin{cases} m^* - 1 & \text{if } nm+1 \text{ is even,} \\ m^* - 2 \text{ or } m^* & \text{if } nm+1 \text{ is odd.} \end{cases}$$

Note that this value does not depend on the actual value of $v_i(0)$ for each i. Therefore we can pick

$$v_i(0) = \begin{cases} \lfloor m/2 \rfloor & \text{if } i \text{ is odd,} \\ \lceil m/2 \rceil & \text{if } i \text{ is even.} \end{cases}$$

This gives a friendly labeling of Amal$((U(n), c), m)$. The fact that c can be adjacent to either a 0-vertex or an 1-vertex on each copy of C_n implies that $0 \leq m^* \leq m$. Hence $m^* - 1 \in \{-1, 0, 1, \ldots, m-1\}$, and $m^* - 2 \in \{-2, -1, 0, 1, \ldots, m-2\}$, thereby proving the theorem. □

Corollary 4.2 *For $n \geq 3$,*

$$BI(U(n)) = \begin{cases} \{0, 1\} & \text{if } n \text{ is odd,} \\ \{0, 1, 2\} & \text{if } n \text{ is even.} \end{cases}$$

Example 13. To describe the friendly labeling of Amal$((U(3), c), m)$, it suffices to list the labels of the three vertices x_1, x_2 and x_3 in each of the m copies of of C_3. The table below shows the labeling for $m = 2, 3, 4$.

C_3	C_3	C_3	C_3	$e(0)$	$e(1)$	$\lvert e(0) - e(1) \rvert$
		110	100	1	1	0
		011	100	2	1	1
		011	001	3	1	2
	011	100	110	2	2	0
	011	001	110	3	2	1
	011	001	011	4	2	2
110	100	110	100	2	2	0
011	100	110	100	3	2	1
011	001	110	100	4	2	2
011	001	011	100	5	2	3
011	001	011	001	6	2	4

When reading this table, recall that x_1 of each cycle is adjacent to c, which is a 0-vertex. See Figure 6 for the labelings for $m = 2$. □

5 Balance Index Sets of Regular Windmills

Definition 5.1. If $x_1, x_2, \ldots, x_n$ are the vertices in the complete graph K_n, we will call the one-point union Amal$((K_n, x_1), m)$ the ***regular windmill graph***. For simplicity we will denote it by WM(n, m).

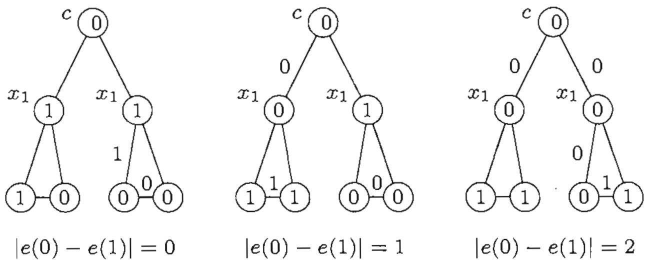

Figure 6: The balance index set of Amal($(U(3), c), 2$).

Benson and Lee [1] investigated the cordialness of WM(n, m).

Theorem 5.1 *For all $n \geq 3$ and all $m \geq 1$, let $p = |V(WM(n,m))| = (n-1)m+1$, then*

$$BI(WM(n,m)) = \begin{cases} \left\{ \dfrac{(n-1)(m-1)}{2} \right\} & \text{if } p \text{ is even,} \\ \left\{ \dfrac{(n-1)(m-2)}{2}, \dfrac{(n-1)m}{2} \right\} & \text{if } p \text{ is odd.} \end{cases}$$

Proof. Call a copy of K_n of type i if it has i vertices other than x_1 that are labeled 0. Restricted to this type i complete graph on n vertices, we have $e(0) = \binom{i+1}{2}$ and $e(1) = \binom{n-i-1}{2}$; hence

$$e(0) - e(1) = \binom{i+1}{2} - \binom{n-i-1}{2} = -\binom{n-1}{2} + (n-1)i.$$

If there are m_i copies of K_n of type i, then over the entire WM(n, m),

$$\begin{aligned} e(0) - e(1) &= -m\binom{n-1}{2} + (n-1)\sum_{i=1}^{n-1} i m_i \\ &= -m\binom{n-1}{2} + (n-1)(v(0)-1). \end{aligned}$$

Since $|V(\mathrm{WM}(n,m))| = p = (n-1)m+1$, we find

$$v(0) - 1 = \begin{cases} \dfrac{(n-2)m+m-1}{2} & \text{if } p \text{ is even,} \\ \dfrac{(n-2)m+m-2}{2} \text{ or } \dfrac{(n-2)m+m}{2} & \text{if } p \text{ is odd.} \end{cases}$$

The proof is now complete. □

Corollary 5.2 *For $n \geq 3$,*

$$BI(K_n) = \begin{cases} \{0\} & \textit{if } n \textit{ is even,} \\ \{(n-1)/2\} & \textit{if } n \textit{ is odd.} \end{cases}$$

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