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Note

A note on edge-graceful spectra of the square of paths

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Abstract

For a simple path P_r on r vertices, the square of P_r is the graph P_r^2 on the same set of vertices of P_r, and where every pair of vertices of distance two or less in P_r is connected by an edge. Given a (p, q)-graph G with p vertices and q edges, and a nonnegative integer k, G is said to be k-edge-graceful if the edges can be labeled bijectively by $k, k+1, \ldots, k+q-1$, so that the induced vertex sums (mod p) are pairwise distinct, where the vertex sum (mod p) at a vertex is the sum of the labels of all edges incident to such a vertex, modulo the number of vertices p. We call the set of all such k the edge-graceful spectrum of G, and denote it by $egI(G)$. In this article, the edge-graceful spectrum $egI(P_r^2)$ for the square of paths P_r^2 is completely determined for odd r.

Keywords: Edge-graceful labeling; Edge-graceful spectrum; Square of paths

1. Introduction and preliminaries

Given an integer $k \geq 0$, a graph $G = (V, E)$ with p vertices and q edges is called k-edge-graceful, if there exists a bijection $f : E \to \{k, k+1, k+2, \ldots, k+q-1\}$ such that the induced vertex sum $f^+ : V \to \mathbb{Z}_p$, where $f^+(u) = \sum\{f(uv) : uv \in E\}$ (mod p), is also a bijection. Such a labeling is called a k-edge-graceful labeling. The study of the class of 1-edge-graceful graphs was initiated by Lo [17], and such a class of graphs are also known as edge-graceful graphs. Edge-graceful labeling can be viewed as a dual concept of the well-known graceful labeling. A good account on graceful labeling problems and other types of graph labeling problems can be found in the dynamic survey of Gallian [2]. Fig. 1 shows that K_4 is k-edge-graceful for $k = 1, 2, 3, 4$. We have the following necessary condition for a graph to be a k-edge-graceful graph, which is a generalization of Lo's condition in [17]: if a (p, q)-graph G is k-edge-graceful, then it satisfies the condition that $q(q + 2k - 1) \equiv \frac{p(p-1)}{2}$ (mod p). A conjecture similar to the well-known graceful tree conjecture is that the above mentioned necessary condition is also sufficient for 1-edge-graceful graphs, particularly restricted to the class of trees [7]. However, the theory of 1-edge-graceful graphs is completely different from that of other k-edge-graceful graphs. For example, trees of order 4 are 2-edge-graceful but not 1-edge-graceful (please see Fig. 2). Some classes of 1-edge-graceful graphs are investigated in [1,3–23]. Certain

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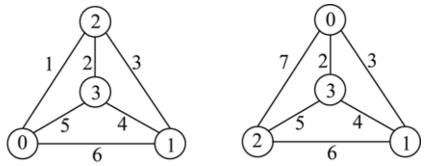

Fig. 1. Various k-edge-gracefulness of K_4.

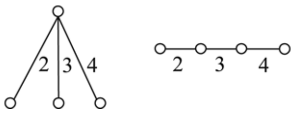

Fig. 2. Trees that are 2-edge-graceful but not 1-edge-graceful.

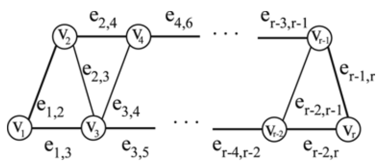

Fig. 3. The edge decomposition of an outer cycle and an inner path in P_r^2.

properties of k-edge-graceful graphs are considered in [4,9,16]. We denote the set of all integers $k \geq 0$ such that G is k-edge-graceful by $egI(G)$. This set is called the edge-graceful spectrum of G. In [4,9], the edge-graceful spectra of wheels and some $(p, p+1)$-graphs are studied.

For a graph G, the square of G is the graph G^2 on the same set of vertices of G, and where every pair of vertices of distance two or less in G is connected by an edge. For the sake of convenience, the graph square of paths P_r^2 can be treated as an edge disjoint union of an inner path with an outer cycle (please see Fig. 3 for an example).

In this paper, we completely determine the edge-graceful spectrum of square of paths, P_r^2, for odd r. Further we investigate the edge-graceful spectrum of square of paths, P_r^2, for some special cases of even r and finally we discuss related problems.

2. Main results

Here we determine the edge-graceful spectrum of square of paths, P_r^2, for odd r. Notice that since we are considering the concept modulo the number of vertices p, therefore it is clear that the calculation of the edge-graceful spectrum of graphs can be reduced to a finite set. We state the fact as a lemma as follows:

Lemma 1. *Let G be a (p, q)-graph with the edge-graceful spectrum $egI(G)$. If $k \in egI(G)$, then $k + t \cdot p \in egI(G)$ for any nonnegative integer t.*

We prove the necessary conditions for a graph to be k-edge-graceful, for the square of paths first:

Theorem 2. *If the graph P_r^2 is k-edge-graceful, then the following conditions on k hold good.*

(1) *For an odd integer $r \geq 3$,*
  (a) *if $3 \mid r$, then $k \equiv 2 \pmod{\frac{r}{3}}$, or*
  (b) *if $3 \nmid r$, then $k \equiv 2 \pmod{r}$.*

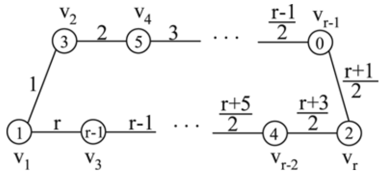

Fig. 4. An edge labeling on the outer cycle of P_r^2 and the associated vertex sums.

(2) *For an even integer* $r \geq 2$,
  (a) (i) *if* $r \equiv 0 \pmod 4$ *and* $3 \mid r$, *then* $k \equiv \frac{r}{12} + 2 \pmod{\frac{r}{6}}$, *or*
  (ii) *if* $r \equiv 0 \pmod 4$ *and* $3 \nmid r$, *then* $k \equiv \frac{r}{4} + 2 \pmod{\frac{r}{2}}$, *or*
  (b) *if* $r \equiv 2 \pmod 4$, *then* $egI(P_r^2) = \emptyset$.

Proof. The necessary condition for a graph to be a k-edge-graceful graph is $q(q + 2k - 1) \equiv \frac{p(p-1)}{2} \pmod p$. Note that the graph P_r^2 has r vertices and $2r - 3$ edges. Therefore, the necessary condition becomes $-3(2k - 4) \equiv r(r - 1)/2 \pmod r$, and the calculations are as follows:

(1) r is odd. Then the necessary condition becomes $3k \equiv 6 \pmod r$.
  (a) If $3 \mid r$, then $k \equiv 2 \pmod{\frac{r}{3}}$.
  (b) If $3 \nmid r$, then $k \equiv 2 \pmod r$.
(2) r is even. Then the necessary condition becomes $6k \equiv \frac{r}{2} + 12 \pmod r$.
  (a) If $r \equiv 0 \pmod 4$, that is, $r = 4n$ for some $n \in \mathbb{Z}$, then the necessary condition is $3k \equiv n + 6 \pmod{2n}$.
    (i) If $3 \mid r$, then $k \equiv \frac{n}{3} + 2 \pmod{\frac{2n}{3}}$. That is, $k \equiv \frac{r}{12} + 2 \pmod{\frac{r}{6}}$.
    (ii) If $3 \nmid r$, then $k \equiv n + 2 \pmod{2n}$. That is, $k \equiv \frac{r}{4} + 2 \pmod{\frac{r}{2}}$.
  (b) If $r \equiv 2 \pmod 4$, that is, $r = 4n + 2$ for some $n \in \mathbb{Z}$. Then the necessary condition becomes $6k \equiv 2n + 13 \pmod{4n + 2}$. But this equation is incompatible by considering the parity. □

We show that the necessary condition in Theorem 2 is also sufficient for the graph P_r^2, where r is odd. To determine the edge-graceful spectrum of P_r^2 for odd r, by Lemma 1 and Theorem 2, we show that

(1) if $3 \nmid r$, then P_r^2 is 2-edge-graceful,
(2) if $3 \mid r$, then P_r^2 is 2-edge-graceful, $(n + 2)$-edge-graceful, and $(2n + 2)$-edge-graceful, where $n = \frac{r}{3}$.

Theorem 3. *P_r^2 is 2-edge-graceful, where r is odd.*

Proof. Let the path P_r be the graph with vertex set $\{v_1, v_2, \ldots, v_r\}$ and the edge set $\{v_i v_{i+1} : 1 \leq i \leq r-1\}$. Suppose that $e_{i,j}$ denotes the edge $v_i v_j \in E(P_r^2)$. Then the outer edges $e_{1,2}$, $e_{r,r-1}$, and $e_{i,i+2}$, where $1 \leq i \leq r - 2$, form an odd cycle, and the inner edges $e_{i,i+1}$, where $2 \leq i \leq r - 2$, form an even path (please see Fig. 3). To show that the graph P_r^2 is 2-edge-graceful, we label the integers $2, 3, \ldots, 2r-2$ to the $2r-3$ edges. Since the set $\{2, 3, \ldots, 2r-2\}$ is congruent (mod r) to the multi-set $\{1, 2, \ldots, r\} \cup \{2, 3, \ldots, r - 2\}$, we assign $1, 2, \ldots, r$ to the r edges of the outer cycle in P_r^2, and $2, 3, \ldots, r - 2$ to the $r - 3$ edges of the inner path in P_r^2 in the following way. We label $1, 2, 3, \ldots, \frac{r-1}{2}, \frac{r+1}{2}, \frac{r+3}{2}, \frac{r+5}{2}, \ldots, r$ to the edges $e_{1,2}, e_{2,4}, e_{4,6}, \ldots, e_{r-3,r-1}, e_{r-1,r}, e_{r-2,r}, e_{r-4,r-2}, \ldots, e_{1,3}$ as in Fig. 4. Therefore the outer odd cycle is edge-graceful on its own and the vertex sums of the vertices $v_1, v_2, v_4, \ldots, v_{r-1}, v_r, v_{r-2}, v_{r-4}, \ldots, v_3$ induced from this labeling are $1, 3, 5, \ldots, r, 2, 4, 6, \ldots, r - 1 \pmod r$.

Then we assign $2, 3, \ldots, r - 2$ to the edges of the inner path in P_r^2, by assigning $2, 4, \ldots, r - 3$ to the edges $e_{2,3}, e_{4,5}, \ldots, e_{r-3,r-2}$, and assigning $3, 5, \ldots, r - 2$ to the edges $e_{r-2,r-1}, e_{r-4,r-3}, \ldots, e_{3,4}$ respectively, as in Fig. 5. Hence the vertex sums of vertices $v_2, v_3, v_4, v_5, \ldots, v_{r-3}, v_{r-2}, v_{r-1}$ induced from this edge labeling are $2, 0, 2, 0, \ldots, 2, 0, 3 \pmod r$. Notice that the vertices $v_2, v_4, \ldots, v_{r-3}, v_{r-1}$ have nonzero contributions, and the vertices $v_3, v_5, \ldots, v_{r-2}$ have zero contributions from the vertex sums given by the edge labeling of the inner path.

Therefore, to show that the vertex sums of the graph P_r^2 are pairwise distinct with the above assigned labels, it suffices to show the invariance of the vertex sums over the vertices $v_2, v_4, \ldots, v_{r-3}, v_{r-1}$. The contribution of vertex sums obtained from the inner path are $2, 2, \ldots, 2, 3$. Hence the vertex sums of the vertices $v_2, v_4, \ldots, v_{r-3}, v_{r-1}$ in P_r^2 are $5, 7, \ldots, r, 3$ (note that $r \equiv 0$), which are exactly the set of vertex sums $3, 5, \ldots, r - 2, r$ restricted on the outer cycle, with the same elements cycling around. Hence the proof is complete. □

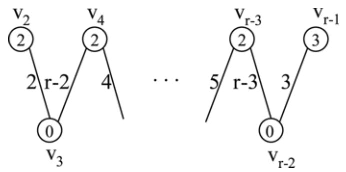

Fig. 5. An edge labeling on the inner path of P_r^2 and the associated vertex sums.

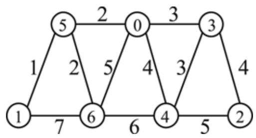

Fig. 6. P_7^2 is 2-edge-graceful.

Please see the Fig. 6 for an example that P_7^2 is 2-edge-graceful. Therefore we have obtained the edge-graceful spectrum of P_r^2, where r is odd and $3 \nmid r$. But if we would like to get the edge-graceful spectrum of P_r^2, where r is odd and $3 \mid r$, we must prove in addition that P_r^2 is also $(n+2)$-edge-graceful and $(2n+2)$-edge-graceful, where $r = 3n$. These statements are proved in the following Theorems 5 and 6. Theorem 4 will be useful for proving Theorem 5.

Theorem 4. *The set* $\{1, 2, \ldots, 3n\} - \{n-1, n, n+1\}$, *where* $n = 3, 5, \ldots$, *can be partitioned into* $(n-1)$ 3-*element subsets such that the sum of the elements in each subset is* $0 \pmod{3n}$.

Proof. If it can be shown that there is such a partition for a set $\{1, 2, \ldots, 3n\}$ which includes the set $\{n-1, n, n+1\}$, then we are done. Hence we divide the problem into three cases as follows:

(1) $n \equiv 0 \pmod 3$.
Assign $S_i = \{3j-1, 3i, 3j+1 : (i \in \{1, 3, 5, \ldots, n\}, i+2j = n) \text{ or } (i \in \{2, 4, 6, \ldots, n-1\}, i+2j = 2n)\}$ with $-1 \equiv 3n-1 \pmod{3n}$. Also $S_{\frac{n}{3}} = \{n-1, n, n+1\}$.

(2) $n \equiv 1 \pmod 3$.
Assign $S_i = \{3i, 3j+1, 3j+2 : (i \in \{2, 4, 6, \ldots, n-1\}, i+2j = n-1) \text{ or } (i \in \{1, 3, 5, \ldots, n\}, i+2j = 2n-1)\}$. Also $S_{\frac{n-1}{3}} = \{n-1, n, n+1\}$.

(3) $n \equiv 2 \pmod 3$.
Assign $S_i = \{3j-2, 3j-1, 3i : (i \in \{2, 4, 6, \ldots, n-1\}, i+2j = n+1) \text{ or } (i \in \{1, 3, 5, \ldots, n\}, i+2j = 2n+1)\}$. Also $S_{\frac{n+1}{3}} = \{n-1, n, n+1\}$.

It is easy to see that $\bigcup S_i = \{1, 2, \ldots, 3n\}$, $S_i \cap S_j = \emptyset$ for all $i \neq j$, and the sum of elements in each S_i is $0 \pmod{3n}$. □

Theorem 5. P_r^2 *is* $(n+2)$-*edge-graceful, where* $r = 3n$ *is odd.*

Proof. As in the proof of Theorem 3, we let the path P_r be a graph with vertex set $\{v_1, v_2, \ldots, v_r\}$, and the edge set $\{v_i v_{i+1} : 1 \leq i \leq r-1\}$. Assume that $e_{i,j}$ denotes the edge $v_i v_j \in E(P_r^2)$. We partition the edges of the graph P_r^2 into the following sets: $A_1 = \{e_{1,2}, e_{1,3}, e_{2,3}, e_{2,4}, e_{3,4}, e_{4,5}\}$, $A_i = \{e_{j,j+2}, e_{j+1,j+3}, e_{j+2,j+3}, e_{j+2,j+4}, e_{j+3,j+4}, e_{j+4,j+5}\}$, where i ranges from 2 to $n-1$ and $j = 3(i-1)$, and $A_n = \{e_{r-3,r-1}, e_{r-2,r}, e_{r-1,r}\}$.

To show that the graph P_r^2 is $(n+2)$-edge-graceful, we must label $n+2, n+3, \ldots, 7n-2$ to these $2r-3$ edges. Note that the set $\{n+2, n+3, \ldots, 7n-2\}$ is congruent to the multi-set $\{n-1, n, n+1\} \cup (\{1, 2, \ldots, n-2\} \cup \{n+2, n+3, \ldots, 3n\}) \cup (\{1, 2, \ldots, n-2\} \cup \{n+2, n+3, \ldots, 3n\})$. The set $\{1, 2, \ldots, n-2\} \cup \{n+2, n+3, \ldots, 3n\}$ can be partitioned into $n-1$ disjoint subsets $a_i = \{x_i, y_i, z_i : x_i + y_i + z_i \equiv 0 \pmod r\}$, where $i = 1, 2, \ldots, n-1$. Let $a_1 = \{x_1 = 3n, y_1, z_1\}$, $a_{n-1} = \{x_{n-1}, y_{n-1}, z_{n-1} = 2n\}$, and $a_n = \{x_n = n, y_n = n+1, z_n = n-1\}$. Then we assign the integers of a_i to the edges of A_i, where i ranges from 1 to n, by labeling the integers $x_1, y_1, z_1, y_1, x_1, z_1$ to the edges $e_{1,2}, e_{1,3}, e_{2,3}, e_{2,4}, e_{3,4}, e_{4,5}$, labeling the integers $x_i, y_i, z_i, y_i, x_i, z_i$ to the edges $e_{j,j+2}, e_{j+1,j+3}, e_{j+2,j+3}, e_{j+2,j+4}, e_{j+3,j+4}, e_{j+4,j+5}$, where $j = 3(i-1)$ and i ranges from 2 to $n-1$, and labeling the integers x_n, y_n, z_n to the edges $e_{n-3,n-1}, e_{n-2,n}, e_{n-1,n}$, respectively.

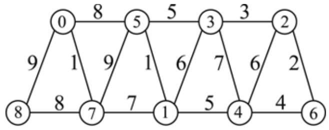

Fig. 7. P_9^2 is 5-edge-graceful.

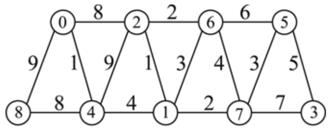

Fig. 8. P_9^2 is 8-edge-graceful.

Therefore the induced vertex sums $f^+ : V(P_r^2) \to \mathbb{Z}_r$, defined by $f^+(u) = \sum\{f(uv) : uv \in E(P_r^2)\} \pmod r$, can be calculated as follows modulo r:

$$f^+(v_1) = x_1 + y_1 \equiv y_1$$
$$f^+(v_2) = x_1 + y_1 + z_1 \equiv 0 \equiv x_1.$$

For all i, $2 \le i \le n-1$, we have:

$$f^+(v_{3(i-1)}) = (x_{i-1} + y_{i-1} + z_{i-1}) + x_i \equiv x_i$$
$$f^+(v_{3(i-1)+1}) = (x_{i-1} + y_{i-1} + z_{i-1}) + y_i \equiv y_i$$
$$f^+(v_{3(i-1)+2}) = (x_i + y_i + z_i) + z_{i-1} \equiv z_{i-1}.$$

The remaining vertex sums are:

$$f^+(v_{3n-3}) = (x_{n-1} + y_{n-1} + z_{n-1}) + x_n \equiv x_n$$
$$f^+(v_{3n-2}) = (x_{n-1} + y_{n-1} + z_{n-1}) + y_n \equiv y_n$$
$$f^+(v_{3n-1}) = (x_n + z_{n-1}) + z_n \equiv z_n$$
$$f^+(v_{3n}) = y_n + z_n \equiv 2n \equiv z_{n-1}.$$

It is clear that all the vertex sums are pairwise distinct, hence we are done. □

Please see the Fig. 7 for an example that P_r^2 is $(n+2)$-edge-graceful, where $r = 3n$ is odd. From the above, we know that P_r^2 is $(n+2)$-edge-graceful, where $r = 3n$ is odd. The $(2n+2)$-edge-graceful labeling is basically obtained by shifting n over the $(n+2)$-edge-graceful labeling. We show below how to get such a $(2n+2)$-edge-graceful labeling.

Theorem 6. *P_r^2 is $(2n+2)$-edge-graceful, where $r = 3n$ is odd.*

Proof. Since P_r^2 is $(n+2)$-edge-graceful, when r is odd and $r = 3n$. We have the disjoint sets $a_i = \{x_i, y_i, z_i : x_i + y_i + z_i \equiv 0 \pmod r\}$, where $i = 1, 2, \ldots, n-1$. The $(2n+2)$-edge-graceful labeling can be made by shifting the $(n+2)$-edge-graceful labeling. Let the partite subsets be $a_i = \{x_i + n, y_i + n, z_i + n\}$, where $(x_i+n)+(y_i+n)+(z_i+n) \equiv 0 \pmod{r = 3n}$, for $i = 1, 2, \ldots, n-1$, and $a_n = \{x_n = 2n, y_n = 2n+1, z_n = 2n-1\}$. Then we denote $a_1 = \{x_1 = 3n, y_1, z_1\}$, $a_{n-1} = \{x_{n-1}, y_{n-1}, z_{n-1} = n\}$. Such assignment in a_i to the edges in A_i, when i ranges from 1 to n, is similar to the one in Theorem 5, hence it is $(2n+2)$-edge-graceful. □

Please see the Fig. 8 for an example that P_r^2 is $(2n+2)$-edge-graceful, where $r = 3n$ is odd. By the above theorems, we completely determine the edge-graceful spectrum of graph P_r^2 for odd r. However the case of r being a multiple of 4 still remains open. We will give the labeling of the first few special cases in the following section.

In this study, we have observed the close connection between the edge-graceful labeling and particular types of integer sequences. In fact, the partition of the finite sets described in the Theorem 4 corresponds to a certain type of

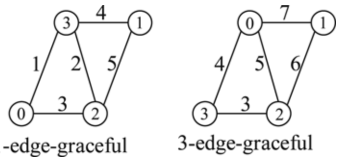

Fig. 9. Edge-gracefulness of P_4^2.

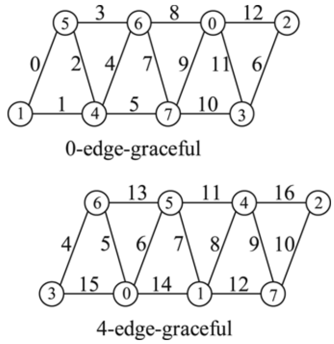

Fig. 10. Edge-gracefulness of P_8^2.

edge partition of the graph P_r^2. Again these facts provide us with more evidences of intrinsic relationships among graph labeling, graph partitioning, and number theoretical properties.

3. Edge-graceful spectra of square of paths on even vertices

Here we study the edge-graceful spectrum of P_r^2, where r is a multiple of 4, with special cases of few vertices.

Theorem 7. $egI(P_4^2) = \{1 + 2t : t = 0, 1, 2, 3, \ldots\}$.

Proof. It suffices to show that P_4^2 is 1-edge-graceful and 3-edge-graceful by the Lemma 1 and Theorem 2. Fig. 9 shows that P_4^2 is 1-edge-graceful and 3-edge-graceful. Therefore we have $1 + 4t$ and $3 + 4t$ as the spectrum for all integers $t > 0$. Thus the edge-graceful-spectrum $egI(P_4^2) = \{1 + 2t : t = 0, 1, 2, 3, \ldots\}$. □

Theorem 8. $egI(P_8^2) = \{8t, 4 + 8t : t = 0$ *or* $t \in \mathbb{N}\}$, *where* $\mathbb{N}$ *is the set of positive integers.*

Proof. The graph P_8^2 has 8 vertices and 13 edges. From the necessary condition for a graph to be a k-edge-graceful graph, we have the equation $(12 + 2k)13 \equiv \frac{8\times 7}{2} \pmod 8 \Rightarrow 4 + 2k \equiv 4 \pmod 8 \Rightarrow k \equiv 0 \pmod 4$. Again by the Lemma 1, it suffices to show that P_8^2 is 0-edge-graceful and 4-edge-graceful, and Fig. 10 provides these two labeling. □

Theorem 9. $egI(P_{12}^2) = \{2t + 1 : t = 0, 1, 2, \ldots\}$.

Proof. Note that by the Lemma 1 and Theorem 2, it suffices to show that P_{12}^2 is k-edge-graceful, where $k = 1, 3, 5, 7, 9, 11$. Fig. 11 provides the required labeling. □

The edge-graceful spectra of the first few P_r^2 are given, where r is a multiple of 4. However the general labeling for the square of paths has still not been found. It is needed for completing the problem of determination of edge-graceful spectra of the square of paths.

4. Conclusion and open problems

In this paper we completely determine the edge-graceful spectrum of P_r^2 for odd r, while the problem of determining the edge-graceful spectrum of P_r^2, where r is a multiple of 4, remains open. Moreover, the problems to determine the edge-graceful spectrum of higher power of paths P_r or other classes of graphs are also interesting and worth exploring further.

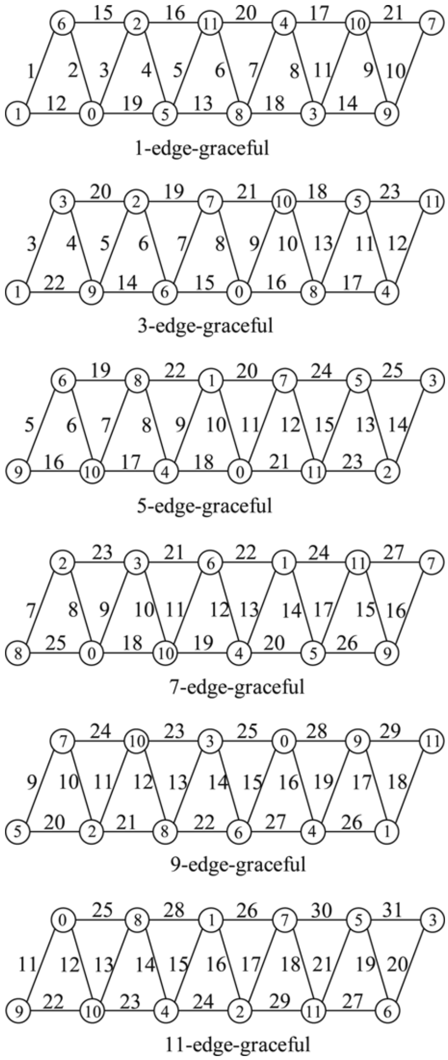

Fig. 11. Edge-gracefulness of P_{12}^2.

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References

[1] S. Cabannis, J. Mitchem, R. Low, On edge-graceful regular graphs and trees, Ars Combinatoria 34 (1992) 129–142.
[2] J.A. Gallain, A dynamic survey of graph labeling, The Electronic Journal of Combinatorics DS6 (2005) 1–148. Ninth Version, January 20.
[3] Jonathan Keene, Andrew Simoson, Balanced strands for asymmetric, edge-graceful spiders, Ars Combinatoria 42 (1996) 49–64.
[4] Kang Qingde, Sin-Min Lee, Wang Ling, On the edge-graceful spectra of the wheel graphs (manuscript).
[5] Q. Kuan, Sin-Min Lee, J. Mitchem, A.K. Wang, On edge-graceful unicyclic graphs, Congressus Numerantium 61 (1988) 65–74.
[6] Li Min Lee, Sin Min Lee, G. Murthy, On edge-graceful labelings of complete graphs — solutions of Lo's conjecture, Congressum Numerantum 62 (1988) 225–233.
[7] Sin-Min Lee, A conjecture on edge-graceful trees, Scientia, Series A 3 (1989) 45–57.
[8] Sin-Min Lee, New directions in the theory of edge-graceful graphs, in: Proceedings of the 6th Caribbean Conference on Combinatorics & Computing, 1991, pp. 216–231.
[9] Sin-Min Lee, Kuo Jie Chen, Yun-Chin Wang, On the edge-graceful spectra of cycles with one chord and dumbbell graphs, Congressus Numerantium 170 (2004) 171–183.

[10] Sin-Min Lee, Peining Ma, Linda Valdes, Siu-Ming Tong, On the edge-graceful grids, Congressus Numerantium 154 (2002) 61–77.
[11] Sin-Min Lee, Eric Seah, Edge-graceful labelings of regular complete k-partite graphs, Congressus Numerantium 75 (1990) 41–50.
[12] Sin-Min Lee, Eric Seah, On edge-gracefulness of the composition of step graphs with null graphs, in: Combinatorics, Algorithms, and Applications in Society for Industrial and Applied Mathematics, 1991, pp. 326–330.
[13] Sin-Min Lee, Eric Seah, On the edge-graceful (n, kn)-multigraphs conjecture, Journal of Combinatorial Mathematics and Combinatorial Computing 9 (1991) 141–147.
[14] Sin-Min Lee, Eric Seah, On edge-gracefulness of kth power cycles, Congressus Numerantium 71 (1990) 237–242.
[15] Sin-Min Lee, E. Seah, P.C. Wang, On edge-gracefulness of the kth power graphs, Bulletin of the Institute of Math, Academia Sinica 18 (1) (1990) 1–11.
[16] Sin-Min Lee, Wang Ling, On k-edge-graceful trees (manuscript).
[17] S.P. Lo, On edge-graceful labelings of graphs, Congressus Numerantium 50 (1985) 231–241.
[18] Jin Peng, W. Li, Edge-gracefulness of $C_m \times C_n$, in: Proceedings of the Sixth Conference of Operations Research Society of China, Changsha, October 10–15, Global-Link Publishing Company, Hong Kong, 2000, pp. 942–948.
[19] J. Mitchem, A. Simoson, On edge-graceful and super-edge-graceful graphs, Ars Combinatoria 37 (1994) 97–111.
[20] A. Riskin, S. Wilson, Edge graceful labellings of disjoint unions of cycles, Bulletin of the Institute of Combinatorics and its Applications 22 (1998) 53–58.
[21] Karl Schaffer, Sin Min Lee, Edge-graceful and edge-magic labelings of Cartesian products of graphs, Congressus Numerantium 141 (1999) 119–134.
[22] W.C. Shiu, P.C.B. Lam, H.L. Cheng, Edge-gracefulness of composition of paths with null graphs, Discrete Mathematics 253 (2002) 63–76.
[23] W.C. Shiu, Sin-Min Lee, K. Schaffer, Some k-fold edge-graceful labelings of $(p, p-1)$-graphs, Journal of Combinatorial Mathematics and Combinatorial Computing 38 (2001) 81–95.