On The Balanced Windmill Graphs

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ABSTRACT

Let G be a graph with vertex set V(G) and edge set E(G), and let A ={0,1}. A labeling f : V(G) → A induces an edge partial labeling f^* : E(G) → A defined by $f^*(xy) = f(x)$, if and only if f(x)=f(y) for each edge xy ∈ E(G). For i ∈ A, let $v_f(i)$ = card{v ∈ V(G) : f(v) = i} and $e_{f^*}(i)$ = card{e ∈ E(G) : $f^*(e)$ = i}. A labeling f of a graph G is said to be friendly if $| v_f(0) - v_f(1) | \leq 1$. If, $| e_f(0) - e_f(1) | \leq 1$ then G is said to be **balanced**. In this paper we prove several families of regular windmill and general windmill graphs are balanced.

1.Introduction.

A labeling problem of graphs which is called cordial graph labeling was introduced by Cahit [2] in 1986. Let G be a graph with vertex set V(G) and edge set E(G). A binary vertex labeling of G is a mapping from V(G) into the set {0,1}. For each vertex labeling f of G, Cahit considered a binary edge labeling f#: E→ {0,1}, defined by f# ({u, v}) = | f(u) – f(v)| for all {u, v} in E(G). Let $V_0^f(G)$ and $V_1^f(G)$ denote the number of elements in V(G) that are labeled by 0 and 1 under the mapping f respectively. Likewise, let $e_0^{f\#}(G)$ and $e_1^{f\#}(G)$ denote the number of elements in E(G) that are labeled by 0 and 1 under the induced function f# respectively. Cahit called a graph **cordial** if it has the following properties:

(i) $|v_0^f(G) - v_1^f(G)| \leq 1$ and

(ii) $|e_0^{f\#}(G) - e_1^{f\#}(G)| \leq 1$.

Several constructions of cordial graphs, in particular, the Cartesian product, composition of graphs and tensor products, are considered in [1, 7, 11, 12, 14, 16, 17, 19, 21,22,24,25]. For some new and unsolved problems, the reader refer to [4,7,8].

Lee, Liu and Tan considered another labeling problem, the ***balanced labeling*** problem [20]. For any binary vertex labeling, a partial edge labeling f^* of G can be defined in the following way. For each edge {u, v} in E(G), where u, v ∈ V(G),

$$f^*(\{u, v\}) = \begin{cases} 0 & \text{if } f(u) = f(v) = 0 \\ 1 & \text{if } ((u) = f(v) = 1 \end{cases}$$

Note that if $f(u) \neq f(v)$, the edge $\{u, v\}$ is not labeled by f^*. Thus f^* is a partial function from $E(G)$ into the set $\{0, 1\}$, and we shall refer f^* as the induced partial function of f. Let $e_0^f(G)$ and $e_1^f(G)$ denote the number of elements in $E(G)$ that are labeled by 0 and 1 under the induced partial function f^* respectively. Hence,

$$v_0^f(G) = |\{u \in V(G) \mid f(u) = 0\}|$$
$$v_1^f(G) = |\{u \in V(G) \mid f(u) = 1\}|$$
$$e_0^{f^*}(G) = |\{\{u, v\} \in E(G) \mid f^*(\{u, v\}) = 0\}|$$
$$e_1^{f^*}(G) = |\{\{u, v\} \in E(G) \mid f^*(\{u, v\}) = 1\}|$$

With this notation, we now introduce the notion of a balanced graph.

Definition 1.1. Let G be a graph. G is a balanced graph, or G is balanced, if there is a binary vertex labeling f of G that satisfies the following conditions:

(i) $|v_0^f(G) - v_1^f(G)| \leq 1$ and

(ii) $|e_0^{f^*}(G) - e_1^{f^*}(G)| \leq 1$.

A graph G is said to be **strongly vertex-balanced** if G is a balanced graph and $v_0^f(G) = v_1^f(G)$. Similarly, a graph G is said to be **strongly edge-balanced** if G is a balanced graph and $e_0^{f^*}(G) = e_1^{f^*}(G)$. If G is both strongly vertex-balanced and strongly edge-balanced, we say that G is **strongly balanced**. (We will omit the superscripts f and f^* when the context is clear).

Example 1. Figure 1 shows that the $BI(G) = \{0,1,2\}$.

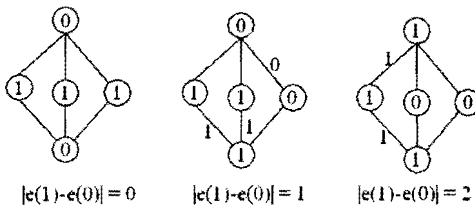


Figure 1.

Example 2. The (6,7)-graph G is balanced.

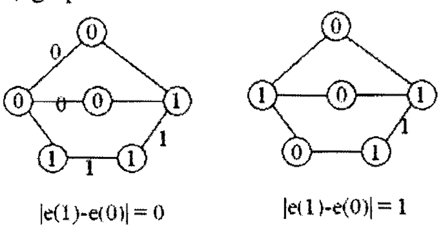


Figure 2.

Example 3. Figure 4 shows that $C_3 x P_3$ is balanced and $C_4 x P_2$ is strongly balanced.

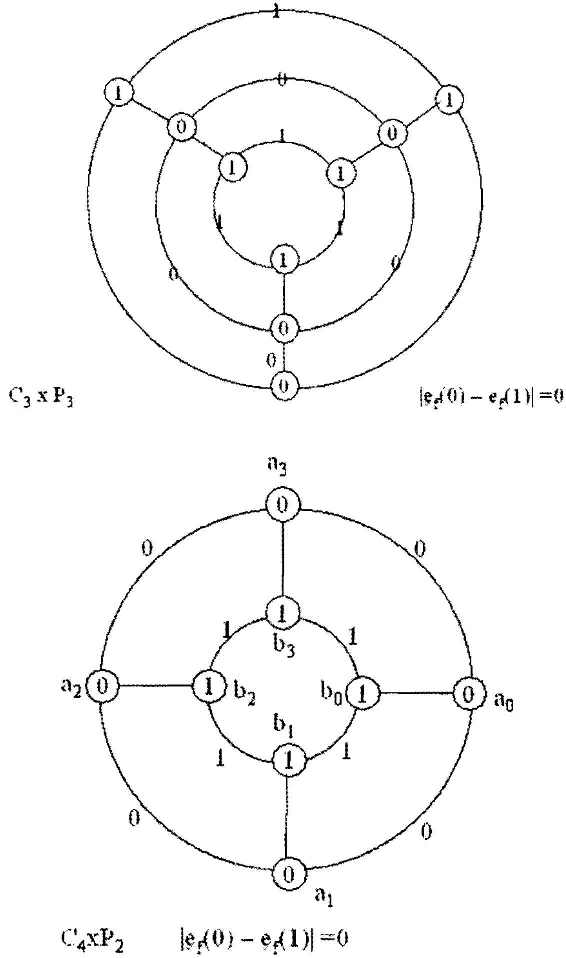


Figure 3.

The following results were established in [20]:

Theorem 1.1 Let G be a k-regular graph with p vertices and q edges,

(i) G is strongly balanced if and only if p is even;

(ii) G is balanced if and only if p is odd and k = 2.

Corollary 1.2 Every cycle C_m is a balanced graph.

Corollary 1.3 For complete graph on n vertices K_m,

(i) K_m is a strongly balanced graph if m is even;

(ii) If m is odd, K_m is balanced if and only if m = 3.

Theorem 1.4 Every path P_m is balanced for $m \geq 1$ and is strongly balanced if m is even.

Theorem 1.5 The complete bipartite graph $K_{m,n}$ is balanced if and only if one of the following conditions holds:
(i) both m and n are even;
(ii) both m and n re odd and $|m-n| \leq 2$;
(iii) one of m and n, say m, is odd, $n = 2t$ and $t = -1, 0$, or $1 \pmod{|m-n|}$.

Suppose a graph G has p vertices and q edges. Assume that p_i of the vertices are of degree r_i, for $i = 1, 2, \ldots, n$, and r_i are integers such that $r_1 < r_2 < \ldots < r_n$. Let

$$S_v = \sum_{i=1}^{n} (p_i - 2a_i) \qquad (1)$$

$$S_e = \sum_{i=1}^{n} (p_i - 2a_i)\, r_i /2 \qquad (2)$$

In [13], a necessary and sufficient condition for a graph to be (strongly) balanced is given. .

Theorem 1.6. Let G be a (p, q)-graph with S_v and S_e as defined in (1) and (2) respectively. G is balanced if and only if there exists a set of integers $\{a_i \mid 0 \leq a_i \leq p_i, i = 1, 2, \ldots, n\}$ such that $|S_v| \leq 1$ and $|S_e| \leq 1$. Furthermore, G is strongly balanced if and only if there exists a set of integers $\{a_i \mid 0 \leq a_i \leq p_i, i = 1, 2, \ldots, n\}$ such that $S_v = 0$, and $S_e = 0$.

Let $K(m_1, m_2, \ldots, m_n)$ be the one-point amalgamation of the complete graphs with $m_1, m_2, \ldots, m_n$ vertices. Call the vertex c at which the complete graphs are amalgamated the **center** of $K(m_1, \ldots m_n)$. If k of the m values are equal to the same value a, and if no confusion could arise, we use a^k to denote these values. Thus $K(a^k)$ is the regular windmill with k component each is a complete graph K_a. The general construction of one-point union was consedidered in [25].

In this paper we investigate which windmill graphs are balanced. We illustrate how the Theorem 1.6 can be used to determine the (strongly) balancedness of several families of windmill graphs. In [1], we consider the cordialness for windmill graphs. For other results of balanced graphs see [12,13,20].

2. Regular Windmill graphs

Theorem 2.1. The regular windmill graph $K(3^k)$ is balanced if and only if k = 2,3.

Proof. Consider the following cases:

Case 1. k is odd, say k=2t+1, where $t \geq 1$.

Then $p(K(3^k)) = 4t+3$. If the graph $K(3^k)$ is balanced then we have $v(0)= 2t+1$ and $v(1)= 2t+2$.

Subcase 1. Assume c has label 0.

For integer $a \geq 1$, if (t-a)'s triangles are labeled by 0, then 2a triangles will have labeled {0,0,1} on the vertices and (t-a+1)'s triangles have labeled {0,1,1} on the vertices.

Thus we see that $e(0) =3(t-a)+2a$ and $e(1) =t-a+1$. Hence $|e(0)-e(1)| = |2t-1|$, and $K(3^k)$ is not balanced if $t \neq 1$, i.e. $k\neq3$.

Subcase 2. Assume c has label 1.

For integer $a \geq 1$, if (t-a)'s triangles are labeled by 1, then 2a-1 triangles will have labeled {0,1,1} on the vertices and (t-a+1)'s triangles have labeled {0,0,1} on the vertices.

Thus we see that $e(0) =t-a+1$ and $e(1) =3(t-a)+2a-1$. Hence $|e(0)-e(1)| = |2t-2|$, and $K(3^k)$ is not balanced if $t \neq 1$, i.e. $k\neq3$.

Case 2. k is even, say k=2t, where $t \geq 1$.

Then $p(K(3^k)) = 4t+1$. If the graph $K(3^k)$ is balanced then we have $v(0)= 2t$ and $v(1)= 2t+1$.

Subcase 1. Assume c has label 0.

For integer $a \geq 1$, if (t-a)'s triangles are labeled by 0, then (2a-1)'s triangles will have labeled {0,0,1} on the vertices and (t-a+1)'s triangles have labeled {0,1,1} on the vertices.

Thus we see that $e(0) =3(t-a)+2a-1=3t-a-1$ and $e(1) =t-a+1$. Hence $|e(0)-e(1)| = |2t-2|$, and $K(3^k)$ is not balanced if $t \neq 1$, i.e. $k\neq2$.

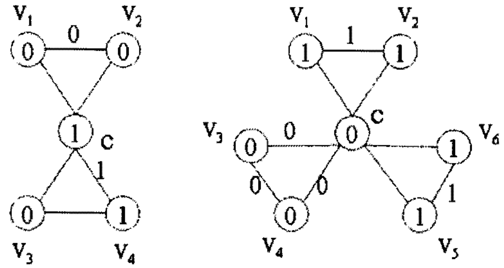
Subcase 2. Assume c has label 1.

For integer $a \geq 1$, if (t-a)'s triangles are labeled by 1, then (t+a)'s triangles will have labeled {0,1,1} on the vertices and (t-a)'s triangles have labeled {0,0,1} on the vertices.

Thus we see that $e(0) =t-a$ and $e(1) =3(t-a)+t+a=4t$.

Hence $|e(0)-e(1)| = |3t+a|$, and $K(3^k)$ is not balanced.

Figure 4 shows that the regular windmill graph $K(3^k)$ is balanced if k = 2,3.□

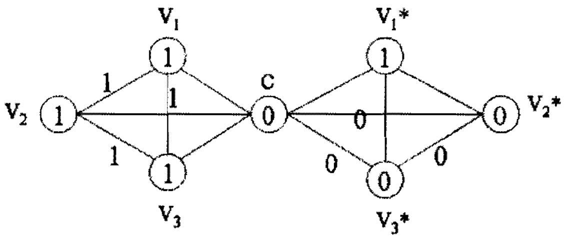


$a_0 = 3,\ a_1 = 0$ $a_0 = 2,\ a_1 = 1$

Figure 4.

Theorem 2.2. The regular windmill graph $K(n^2)$ is balanced for all $n \geq 4$.
Proof. We have two components $\{c, v_1, v_2, \ldots, v_{2m-1}\}$ and $\{c, v_1^*, v_2^*, \ldots, v_{2m-1}^*\}$. If we label $\{c, v_1, v_2, \ldots, v_{2m-1}\}$ by 1 and except c by 0, and we label $\{c, v_1^*, v_2^*, \ldots, v_{2m-1}^*\}$ by 0 except v_1^* by 1. We see that $e(0)=e(1)$. Hence $K(n^2)$ is balanced.□

Example 4. Figure 5 shows that the regular windmill graph $K(4^2)$ is balanced



$K(4^2)$

Figure 5.

Conjecture . If $n \geq 4$, the regular windmill graph $K(n^k)$ is not balanced for all $k \geq 3$.

3. Balanced Windmill graphs with two different types of components.

Theorem 3.1. The windmill graph $K(2^k, 3)$ is balanced if and only if $k = 1,2,3,4$.
Proof.

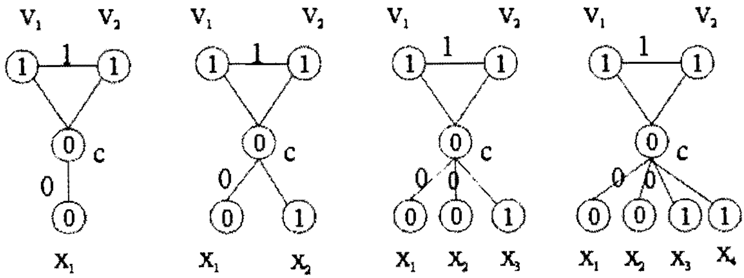


Figure 6.

Now suppose $k > 4$. We want to show that $K(2^k, 3)$ is not balanced. Without loss of generality, assume c has label 0. Consider the following cases:

Case 1. k is odd, say $k=2t+1$, where $t > 1$.
Then $p(K(2^k, 3)) = 2t+4$. Thus we have $v(0)= t+2 =v(1)$.

If $t+1$'s 0 are label on the leave vertices x_i, then we see that $e(0) =t+1$ and $e(1) =1$. Thus $|e(0)-e(1)| = t >1$, and $K(2^k, 3)$ is not balanced.

If t's 0 are label on the leave vertices x_i, then we see that $e(0) =t+1$ and $e(1) =0$. Thus $|e(0)-e(1)| = t +1> 2$, and $K(2^k, 3)$ is not balanced.

If $t-1$'s 0 are label on the leave vertices x_i, then we see that $e(0) =t+2$ and $e(1) =0$. Thus $|e(0)-e(1)| = t +2> 3$, and $K(2^k, 3)$ is not balanced.

Case 2. k is even, say $k=2t$, where $t \geq 3$.
Then $p(K(2^k, 3)) = 2t+3$. Thus we may assume $v(0)= t+1$ and $v(1)=t+2$.

If $t+1$'s 0 are label on the leave vertices x_i, then we see that $e(0) =0$ and $e(1) =t+2$. Thus $|e(0)-e(1)| = t+2>4$.

If t's 0 are label on the leave vertices x_i and c is labeled with 0, then we see that $e(0) =t$ and $e(1) =1$. Thus $|e(0)-e(1)| = t -1\geq 2$.

If $t-1$'s 0 are label on the leave vertices x_i and c is labeled with 0, one of v_1 or v_2 is labeled with 0, then we see that $e(0) =t$ and $e(1) =0$. Thus $|e(0)-e(1)| = t \geq 3$.

If $t-2$'s 0 are label on the leave vertices x_i and $\{v1 , v2 , c\}$ are labeled with 0, then we see that $e(0) =t +1$ and $e(1) =0$. Thus $|e(0)-e(1)| = t+1 \geq 4$.

Thus $K(2^k, 3)$ is not balanced.□

<u>**Theorem 3.2.**</u> The windmill graph $K(2^k, 4)$ is balanced if and only if k = 1,2,3,4,5,6,7.

Proof. Consider the following cases:

<u>**Case 1**</u>. k is odd, say $k=2t+1$, where $t \geq 0$.
Then $p(K(2^k, 4)) = 2t+5$. If $K(2^k, 4)$ is balanced then we have $v(0)= t+2$, $v(1)= t+3$.

<u>**Subcase 1**</u>. Assume c has label 0.

If $t+1$'s 0 are label on the leave vertices x_i, then we see that $e(0) =t+1$ and $e(1) =3$. Thus $|e(0)-e(1)| = |t -2|$, and $K(2^k, 4)$ is not balanced if $t \neq 1,2, 3$ i.e. $k\neq 3,5,7$.

If t's 0 are label on the leave vertices x_i, then we see that e(0) =t+1 and e(1) =1. Thus | e(0)-e(1)| = |t|, and K(2^k, 5) is not balanced if $t \neq 0,1$ i.e. $k \neq 1, 3$.

If t-1's 0 are label on the leave vertices x_i, then we see that e(0) =t+1 and e(1) =0. Thus | e(0)-e(1)| = t +1, and K(2^k, 4) is not balanced if $t \neq 0$ i.e. $k \neq 1$.

Subcase 2. Assume c has label 1.

If t+2's 0 are label on the leave vertices x_i, then we see that e(1) =t+5 and e(0) =0. Thus | e(0)-e(1)| = |t+5| and K(2^k, 4) is not balanced.

If t+1's 0 are label on the leave vertices x_i, then we see that e(0) =1 and e(1) = t+3. Thus | e(0)-e(1)| = |t+3| and K(2^k, 4) is not balanced.

If t's 0 are label on the leave vertices x_i, then we see that e(0) = 1 and e(1) = t+2. Thus | e(0)-e(1)| = t+1 and K(2^k, 4) is not balanced if $t \neq 0$ i.e. $k \neq 1$.

If t-1's 0 are label on the leave vertices x_i, then we see that e(0) =3 and e(1) = t+2. Thus | e(0)-e(1)| = |t-1| and K(2^k, 4) is not balanced if $t \neq 0, 1,2$, i.e. $k \neq 1,3,5$.

Case 2. k is even, say k=2t.

Then p(K(2^k, 4)) = 2t+4. Thus v(0)= t+2 =v(1). Without loss of generality assume c has label 0.

If t+1's 0 are label on the leave vertices x_i, then we see that e(0) =t+1 and e(1) =3. Thus | e(0)-e(1)| = |t-2| and K(2^k, 4) is not balanced if $t \neq 1,2,3$, i.e. $k \neq 2,4,6$.

If t's 0 are label on the leave vertices x_i and c is labeled with 0, then we see that e(0) =t+1 and e(1) =1. Thus | e(0)-e(1)| = t and K(2^k, 4) is not balanced if $t \neq 1$, i.e. $k \neq 2$.

If t-1's 0 are label on the leave vertices x_i and c is labeled with 0, then we see that e(0) =t+2 and e(1) =0. Thus | e(0)-e(1)| = t+2 and K(2^k, 4) is not balanced.

If t-2's 0 are label on the leave vertices x_i, then we see that e(0) =t +4 and e(1) =0. Thus | e(0)-e(1)| = t+4 and K(2^k, 4) is not balanced.

Figure 7 shows that the windmill graph K(2^k, 4) is balanced if k = 1,2,3, 4,5,6,7.

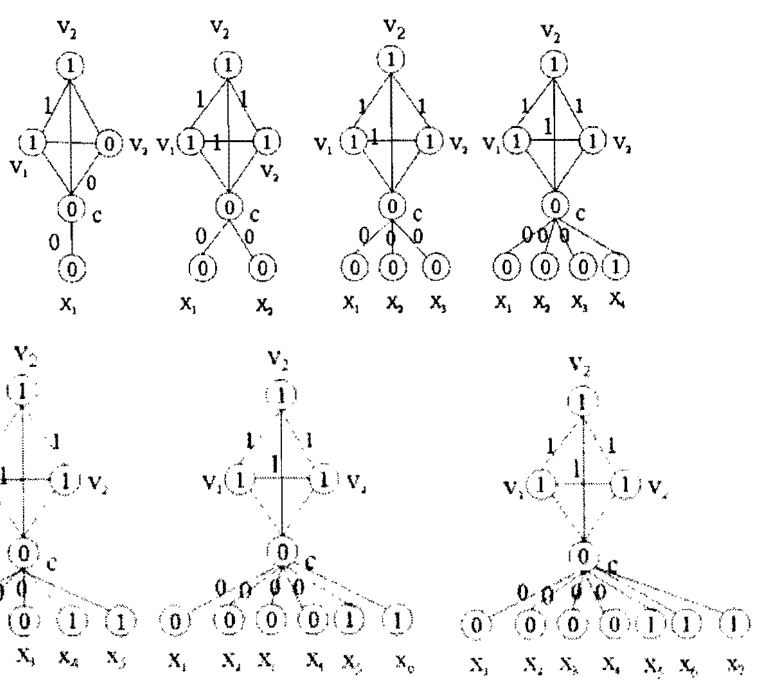


Figure 7.

<u>Theorem 3.3.</u> The windmill graph $K(2^k, 5)$ is balanced if and only if k = 1,2,3,4,5,6,7,8,9,10,11,12.

Proof. Consider the following cases:

<u>Case 1</u>. k is odd, say $k=2t+1$, where $t > 1$.

Then $p(K(2^k, 5)) = 2t+6$. Thus we have $v(0)= t+3 =v(1)$. Without loss of generality, assume c has label 0.

If t+2's 0 are label on the leave vertices x_i, then we see that $e(0) =t+2$ and $e(1) =6$. Thus $| e(0)-e(1)| = | t-4|$, and $K(2^k, 5)$ is not balanced if $t \neq 3,4,5$, i.e. $k\neq 7,9,11$.

If t+1's 0 are label on the leave vertices x_i, then we see that $e(0) =t+2$ and $e(1) =3$. Thus $| e(0)-e(1)| = |t -1|$, and $K(2^k, 5)$ is not balanced if $t \neq 0,1,2$, i.e. $k\neq 1,3,5$.

If t's 0 are label on the leave vertices x_i, then we see that $e(0) =t+3$ and $e(1) =1$. Thus $| e(0)-e(1)| = |t+2|$, and $K(2^k, 5)$ is not balanced.

If t-1's 0 are label on the leave vertices x_i, then we see that $e(0) =t+5$ and $e(1) =0$. Thus $| e(0)-e(1)| = t +5> 1$, and $K(2^k, 5)$ is not balanced.

<u>Case 2</u>. k is even, say $k=2t$.

Then $p(K(2^k, 5)) = 2t+5$. Thus $v(0)= t+2$, $v(1)=t+3$.

Subcase 1. Assume c has label 0.

If t+1's 0 are label on the leave vertices x_i, then we see that e(0) =t+1 and e(1) =6. Thus | e(0)-e(1)| = |t-5| and K(2^k, 5) is not balanced if t ≠4,5,6, i.e. k≠8,10,12.

If t's 0 are label on the leave vertices x_i and c is labeled with 0, then we see that e(0) =t+3 and e(1) =3. Thus | e(0)-e(1)| = t and K(2^k, 5) is not balanced if t ≠ 1, i.e. k≠2.

If t-1's 0 are label on the leave vertices x_i and c is labeled with 0, then we see that e(0) =t+5 and e(1) =0. Thus | e(0)-e(1)| = t+5 and K(2^k, 5) is not balanced.

If t-2's 0 are label on the leave vertices x_i and {v1 , v2 , c} are labeled with 0, then we see that e(0) =t +8 and e(1) =0. Thus | e(0)-e(1)| = t+8 ≥ 4 and K(2^k, 5) is not balanced.

Subcase 2. Assume c has label 1.

If t+1's 0 are label on the leave vertices x_i, then we see that e(0) =t+5 and e(1) =0. Thus | e(0)-e(1)| = |t+5| and K(2^k, 5) is not balanced .

If t's 0 are label on the leave vertices x_i and c is labeled with 0, then we see that e(0) = 1 and e(1) = t+3. Thus | e(0)-e(1)| = t+2 and K(2^k, 5) is not balanced.

If t-1's 0 are label on the leave vertices x_i, then we see that e(0) =3 and e(1) = t+2. Thus | e(0)-e(1)| = |t-1| and K(2^k, 5) is not balanced if t ≠ 1,2, i.e. k≠2,4.

If t-2's 0 are label on the leave vertices x_i, then we see that e(0) = 6 and e(1) =t+2. Thus | e(0)-e(1)| = |t-4| and K(2^k, 5) is not balanced if t ≠ 3,4,5 i.e. k≠6,8,10.

Table of Figure 8 shows that K(2^k, 5) admits balance labeling for k = 1,2,3, 4,5, 6,7,8, 9, 10,11,12. □

n	c	v_1	v_2	v_3	v_4	$\{x_1,x_2,\ldots,x_n\}$
1	0	0	1	1	1	0
2	0	0	1	1	1	0 ,1
3	0	0	1	1	1	0,0,1
4	0	0	1	1	1	0,0,1,1
5	0	0	1	1	1	0, 0,0,1,1
6	0	0	1	1	1	0, 0,0,1,1,1
7	0	1	1	1	1	0, 0,0,0,0,1,1
8	0	1	1	1	1	0, 0,0,0,0,1,1,1
9	0	1	1	1	1	0, 0, 0,0,0,0,1,1,1
10	1	0	0	0	0	0,0,0, 1,1,1,1,1,1,1
11	0	1	1	1	1	0,0,0,0,0,0,0,1,1,1,1
12	0	1	1	1	1	0,0,0,0,0,0,0,1,1,1,1,1

Figure 8.

Theorem 3.4. The windmill graph $K(2^k, 6)$ is balanced if and only if k = 1,2,3.5,6,7,8,9,10,11,14,15,16,17,18,19.

Proof. Consider the following cases:

Case 1. k is odd, say k=2t+1, where t > 1.

Then $p(K(2^k, 6)) = 2t+7$. Thus we have v(0)= t+3 and v(1)= t+4.

Subcase 1. Assume c has label 0.

If t+2's 0 are label on the leave vertices x_i, then we see that e(0) =t+2 and e(1) =10. Thus $|e(0)-e(1)| = |t-8|$, and $K(2^k, 6)$ is not balanced if $t \neq 7,8,9$, i.e. $k \neq 15,17,19$.

If t+1's 0 are label on the leave vertices x_i, then we see that e(0) =t+2 and e(1) =6. Thus $|e(0)-e(1)| = |t-4|$, and $K(2^k, 6)$ is not balanced if $t \neq 3,4,5$, i.e. $k \neq 7,9,11$.

If t's 0 are label on the leave vertices x_i, then we see that e(0) =t+3 and e(1) =3. Thus $|e(0)-e(1)| = |t|$, and $K(2^k, 6)$ is not balanced if $t \neq 0,1$, i.e. $k \neq 1,3$.

If t-1's 0 are label on the leave vertices x_i, then we see that e(0) =t+5 and e(1) =1. Thus $|e(0)-e(1)| = t+4 > 1$, and $K(2^k, 6)$ is not balanced.

Subcase 2. Assume c has label 1.

If t+2's 0 are label on the leave vertices x_i, then we see that e(0) =0 and e(1) =t+9. Thus $|e(0)-e(1)| = |t+9|$, and $K(2^k, 6)$ is not balanced.

If t+1's 0 are label on the leave vertices x_i, then we see that e(0) =1 and e(1) =t+6. Thus $|e(0)-e(1)| = |t+5|$, and $K(2^k, 6)$ is not balanced.

If t's 0 are label on the leave vertices x_i, then we see that e(0) = 3 and e(1) = t+4. Thus $|e(0)-e(1)| = |t+1|$, and $K(2^k, 6)$ is not balanced if $t \neq 0$, i.e. $k \neq 1$.

If t-1's 0 are label on the leave vertices x_i, then we see that e(0) =6 and e(1) =t+3. Thus $|e(0)-e(1)| = |t-3|$ and $K(2^k, 6)$ is not balanced if $t \neq 2,3,4$, i.e. $k \neq 5,7,9$.

Case 2. k is even, say k=2t, where $t \geq 1$.

Then $p(K(2^k, 6)) = 2t+6$. Thus v(0)= t+3= v(1). Without loss of generality, assume c has label 0.

If t+2's 0 are label on the leave vertices x_i, then we see that e(0) =t+2 and e(1) =10. Thus $|e(0)-e(1)| = |t-8|$ and $K(2^k, 6)$ is not balanced if $t \neq 7,8,9$, i.e. $k \neq 14,16,18$.

If t+1's 0 are label on the leave vertices x_i, then we see that e(0) =t+2 and e(1) =6. Thus $|e(0)-e(1)| = |t-4|$ and $K(2^k, 6)$ is not balanced if $t \neq 3,4,5$, i.e. $k \neq 6,8,10$.

If t's 0 are label on the leave vertices x_i and c is labeled with 0, then we see that e(0) =t+3 and e(1) =3. Thus $|e(0)-e(1)| = t$ and $K(2^k, 6)$ is not balanced if $t \neq 1$, i.e. $k \neq 2$.

If t-1's 0 are label on the leave vertices x_i and c is labeled with 0, then we see that e(0) =t+5 and e(1) =1. Thus $|e(0)-e(1)| = t+4$ and $K(2^k, 6)$ is not balanced.

If t-2's 0 are label on the leave vertices x_i and {v1 , v2 , c} are labeled with 0, then we see that e(0) =t +8 and e(1) =0. Thus | e(0)-e(1)| = t+8 ≥ 4 and K(2^k, 6) is not balanced.

Figure 9 shows that the windmill graph K(2^k, 6) is balanced if k = 1,2,3.

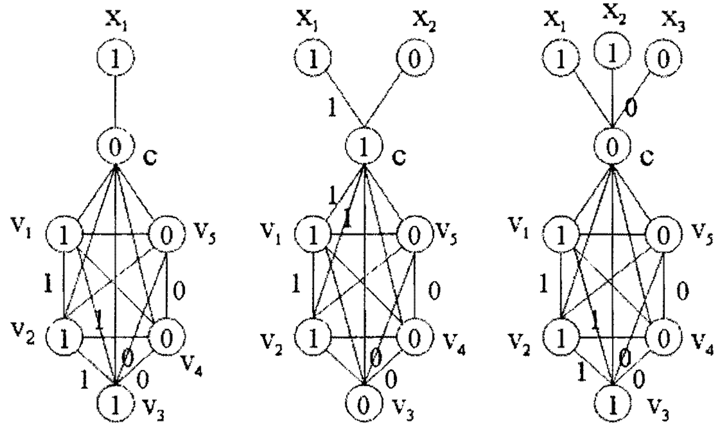


Figure 9.

Table of Figure 10 shows the windmill graph K(2^k, 6) is balanced for k = 5,6,7,8,9,10,11,14,15,16,17,18,19. □

n	c	v_1	v_2	v_3	v_4	v_5	$\{x_1,x_2,...,x_n\}$
5	1	0	0	0	0	1	0, 1,1,1,1
6	0	0	1	1	1	1	0, 0,0,0,1,1
7	1	0	1	1	1	1	0, 0,1,1,1,1,1
8	0	0	1	1	1	1	0, 0,0,0,0,1,1,1
9	1	0	1	1	1	1	0,0,0,1,1,1,1,1,1
10	0	0	1	1	1	1	0,0,0,0,0,0,1,1,1,1
11	0	0	1	1	1	1	0,0,0,0,0,0,1,1,1,1,1
14	0	1	1	1	1	1	0,0,0,0,0,0,0,0,0,1,1,1,1,1
15	0	1	1	1	1	1	0,0,0,0,0,0,0,0,0,1,1,1,1,1,1
16	0	1	1	1	1	1	0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,1
17	0	1	1	1	1	1	0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1
18	0	0	1	1	1	1	0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1
19	0	1	1	1	1	1	0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1

Figure 10.

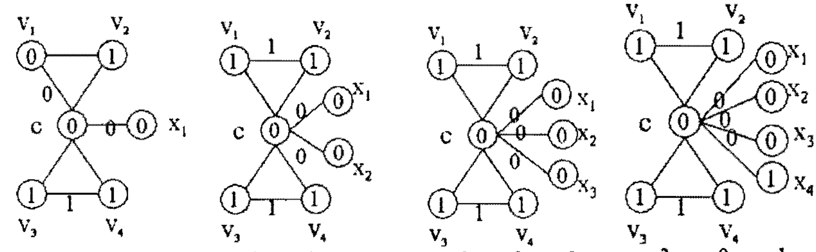
<u>Theorem 3.5.</u> The windmill graph K(2^k, 3^2) is balanced if and only if k = 1,2,3,4.

Proof. Consider the following cases:

Case 1. k is odd, say k=2t+1, where $t \geq 0$.
Then $p(K(2^k, 3^2)) = 2t+6$. Thus we have $v(0)= t+3 = v(1)$.
Assume c has label 0.

If t+2's 0 are label on the leave vertices x_i, then we see that $e(0) = t+2$ and $e(1) = 2$. Thus $|e(0)-e(1)| = |t|$, and $K(2^k, 3^2)$ is not balanced if $t \neq 0,1$, i.e. $k \neq 1,3$.
If t+1's 0 are label on the leave vertices x_i, then we see that $e(0) = t+2$ and $e(1) = 1$. Thus $|e(0)-e(1)| = |t+1|$, and $K(2^k, 3^2)$ is not balanced if $t \neq 0$, i.e. $k \neq 1$.
If t's 0 are label on the leave vertices x_i, then we see that $e(0) = t+3$ and $e(1) = 1$. Thus $|e(0)-e(1)| = |t+2|$, and $K(2^k, 3^2)$ is not balanced.
If t-1's 0 are label on the leave vertices x_i, then we see that $e(0) = t+3$ and $e(1) = 0$. Thus $|e(0)-e(1)| = t+3$, and $K(2^k, 3^2)$ is not balanced.

Case 2. k is even, say k=2t, where $t \geq 1$.
Then $p(K(2^k, 3^2)) = 2t+5$. Thus $v(0)= t+2$ and $= v(1)= t+3$.
Subcase 1. Assume c has label 0.
If t+1's 0 are label on the leave vertices x_i, then we see that $e(0) = t+1$ and $e(1) = 2$. Thus $|e(0)-e(1)| = |t-1|$ and $K(2^k, 3^2)$ is not balanced if $t \neq 1,2$, i.e. $k \neq 2,4$.
If t's 0 are label on the leave vertices x_i, then we see that $e(0) = t+1$ and $e(1) = 1$. Thus $|e(0)-e(1)| = t$ and $K(2^k, 3^2)$ is not balanced if $t \neq 1$, i.e. $k \neq 2$.
If t-1's 0 are label on the leave vertices x_i, then we see that $e(0) = t+2$ and $e(1) = 1$. Thus $|e(0)-e(1)| = t+1$ and $K(2^k, 3^2)$ is not balanced.
If t-2's 0 are label on the leave vertices x_i, then we see that $e(0) = t+2$ and $e(1) = 0$. Thus $|e(0)-e(1)| = t+2$ and $K(2^k, 3^2)$ is not balanced.
Subcase 2. Assume c has label 1.
If t+2's 0 are label on the leave vertices x_i, then we see that $e(0) = 0$ and $e(1) = t+4$. Thus $|e(0)-e(1)| = t+4$ and $K(2^k, 3^2)$ is not balanced.
If t+1's 0 are label on the leave vertices x_i, then we see that $e(0) = 0$ and $e(1) = t+3$. Thus $|e(0)-e(1)| = t+3$ and $K(2^k, 3^2)$ is not balanced.
If t's 0 are label on the leave vertices x_i, then either two 0's are in the same triangle or each triangle contains only one 0.
For the former case, we have $e(0) = 1$ and $e(1) = t+3$.
For the later case, we have $e(0) = 0$ and $e(1) = t+2$.
Both cases imply $|e(0)-e(1)| = t+2$ and $K(2^k, 3^2)$ is not balanced.
If t-1's 0 are label on the leave vertices x_i, then we see that $e(0) = 1$ and $e(1) = t+2$. Thus $|e(0)-e(1)| = t+1$ and $K(2^k, 3^2)$ is not balanced.
If t-2's 0 are label on the leave vertices x_i, then we see that $e(0) = 1$ and $e(1) = t+2$. Thus $|e(0)-e(1)| = t$ and $K(2^k, 3^2)$ is not balanced if $t \neq 1$, i.e. $k \neq 2$.
Figure 11 shows that the windmill graph $K(2^k, 3^2)$ is balanced if k = 1,2,3,4.



a_0=1, a_1=1, a_2=1, a_0=2, a_1=0, a_2=1, a_0=3, a_1=0, a_2=1, a_0=3, a_1=0, a_2=1.

Figure 11.

Using the similar argument as the above proof we can show that

__**Theorem 3.6.**__ The windmill graph $K(2^k, 3^3)$ is balanced if and only if k= 2,4.

__**Example 5.**__ Figure 12 shows that the windmill graphs $K(2^2, 3^3)$ and $K(2^4, 3^3)$ are balanced.

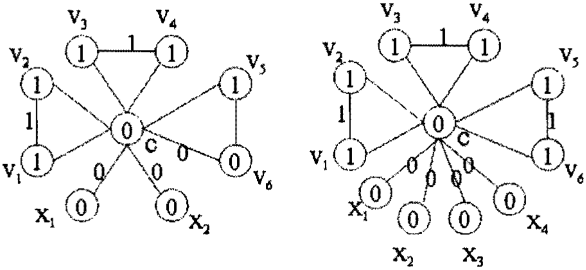


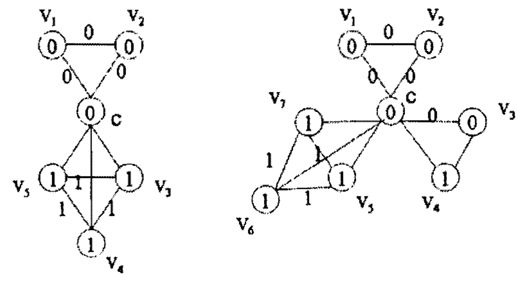
Figure 12.

__**Theorem 3.7.**__ For $n \geq 3$, the windmill graph $K(n^k, n+1)$ is balanced if and only if (1) k= 1 for all n and

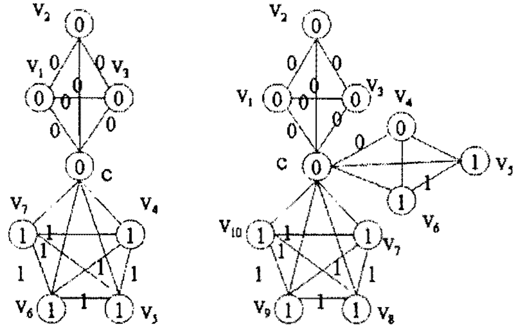
(2) k=2 if n=3 and 4.

Proof. The windmill graph $K(n^k, n+1)$ is balanced if k= 1, for we label all the vertices in K(n) component by 0 and all the vertices of K(n+1) except the root by 1, we see that $|e(0)-e(1)| =0$.

The windmill graph $K(n^2, n+1)$ is balanced if n =3 and 4 (see Figure 13).



a_0= 2, a_1= 0, a_2= 1 a_0= 3, a_1= 0, a_2= 1



$a_0 = 3,\ a_1 = 0,\ a_2 = 1$ $a_0 = 4,\ a_1 = 0,\ a_2 = 1$

Figure 13.

We leave the proof that $K(n^2, n+1)$ is not balanced if $n \geq 5$ for the reader.

<u>**Theorem 3.8.**</u> The windmill graph $K(3^k, 5)$ is balanced if and only if $k \leq 5$.
Proof. Figure 14 shows that $K(3^k, 5)$ is balanced if $k \leq 5$.

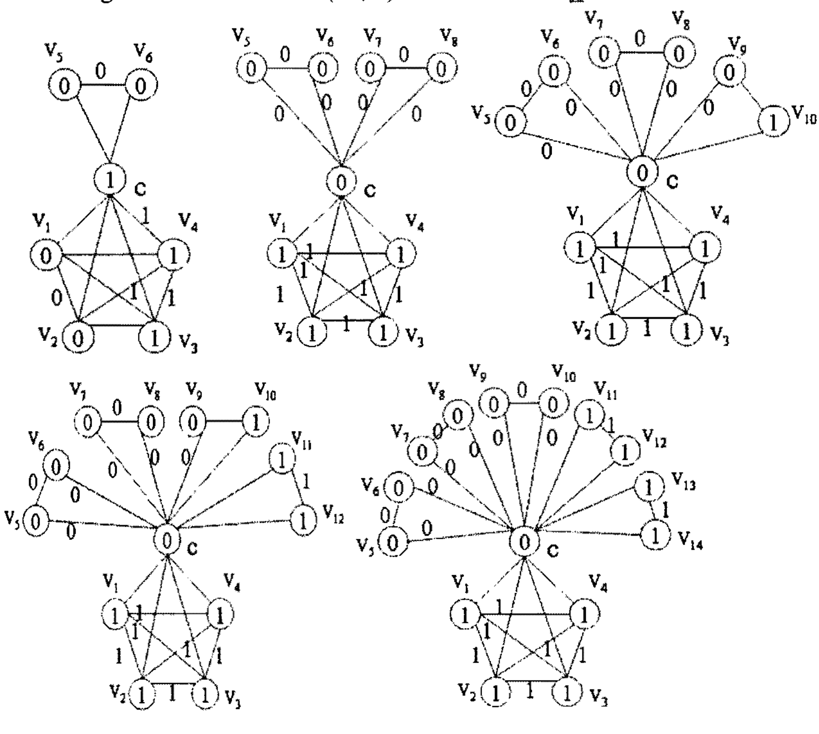


Figure 14.

We leave the proof that $K(3^k, 5)$ is not balanced if $n \geq 6$ for the reader.

Theorem 3.9. The windmill graph $K(3^k, 6)$ is balanced if and only if $k = 1,2,3,4,5,6$ and 7.

Proof. Figure 15 shows that the windmill graph $K(3^k, 6)$ is balanced if $k = 1,2,3,4,5,6$ and 7.

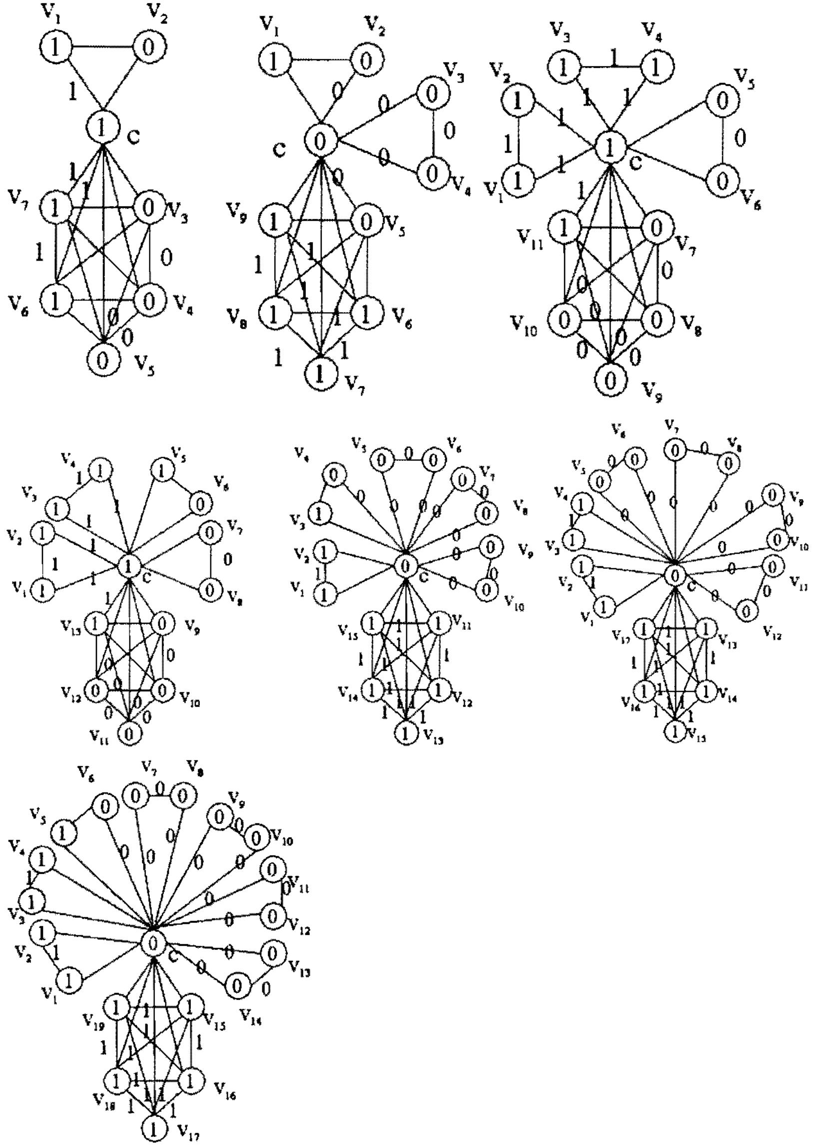


Figure 15.

We leave the proof that $K(3^k, 6)$ is not balanced if $n \geq 8$ for the reader.

4. Balanced Windmill graphs with three different types of components.

In this section we consider windmill graphs with three different types of components.

Theorem 4.1. The windmill graph $K(2^k, 3,4)$ is balanced if and only 1,2,3,4,5,6,7,8.

Proof. Figure 16 shows that the windmill graph $K(2^k, 3,4)$ is balanced if k = 1,2,3

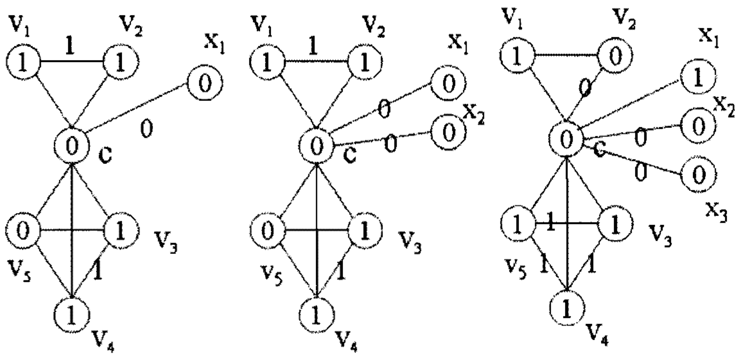


Figure 16.

Table of Figure 17 shows that the windmill graph $K(2^k, 3,4)$ is balanced for k = 1,2,3,4,5,6,7,8.

n	c	v_1	v_2	v_3	v_4	v_5	$\{x_1,x_2,\ldots,x_n\}$
1	0	1	1	1	1	0	0
2	0	1	1	1	1	0	0,0
3	0	1	0	1	1	1	1,0,0
4	0	1	0	1	1	1	1,0,0,0
5	0	1	1	1	1	1	1, 0,0,0,0
6	0	1	1	1	1	1	1, 1,0,0,0,0
7	0	1	1	1	1	1	1,1,0,0,0,0,0
8	0	1	1	1	1	1	1,1,0,0,0,0,0,0

Figure 17.

We want to show that if $k \geq 9$, the windmill graph $K(2^k, 3,4)$ is not balanced. Consider the following cases:

Case 1. k is odd, say k=2t+1, where $t \geq 0$.

Then $p(K(2^k, 3,4)) = 2t+7$. Thus we have v(0)= t+3 , v(1) =t+4.

Subcase 1. Assume c has label 0.

If t+2's 0 are label on the leave vertices x_i, then we see that e(0) =t+2 and e(1) =4. Thus $|e(0)-e(1)| = |t-2|$, and $K(2^k, 3,4)$ is not balanced if $t \neq 1,2,3$, i.e. $k \neq 3,5,7$.

If t+1's 0 are label on the leave vertices x_i, then

(1) if one of $\{v_1,v_2\}$ is labeled by 0, we see that e(0) =t+2 and e(1) =3.

(2) if one of $\{v_3,v_4,v_5\}$ is labeled by 0, we see that e(0) =t+2 and e(1) =2.

Thus $|e(0)-e(1)| = |t-1|$, and and $K(2^k, 3,4)$ is not balanced if $t \neq 0,1,2$, i.e. $k\neq 1,3,5$.

If t's 0 are label on the leave vertices x_i, then

(1) if $\{v_1,v_2\}$ are labeled by 0, we see that e(0) =t+3 and e(1) =3. Thus $|e(0)-e(1)| = |t|$, and $K(2^k, 3,4)$ is not balanced if $t \neq 0,1$, i.e. $k\neq 1,3$.

(2) if one of $\{v_1,v_2\}$ is labeled by 0 and one of $\{v_3,v_4,v_5\}$ is labeled by 0, we see that e(0) =t+2 and e(1) =1. Thus $|e(0)-e(1)| = |t+1|$, and $K(2^k, 3,4)$ is not balanced if $t \neq 0$, i.e. $k\neq 1$.

(3) if two of $\{v_3,v_4,v_5\}$ are labeled by 0, we see that e(0) =t+3 and e(1) =1. Thus $|e(0)-e(1)| = |t+2|$, and $K(2^k, 3,4)$ is not balanced .

If t-1's 0 are label on the leave vertices x_i, then

(1) if two of $\{v_1,v_2\}$ are labeled by 0 and one of $\{v_3,v_4,v_5\}$ is labeled by 0, we see that e(0) =t+3 and e(1) =1. Thus $|e(0)-e(1)| = |t+2|$, and $K(2^k, 3,4)$ is not balanced.

(2) if one of $\{v_1,v_2\}$ is labeled by 0 and two of $\{v_3,v_4,v_5\}$ are labeled by 0, we see that e(0) =t+3 and e(1) =0. Thus $|e(0)-e(1)| = |t+3|$, and $K(2^k, 3,4)$ is not balanced.

(3) if all of $\{v_3,v_4,v_5\}$ are labeled by 0, we see that e(0) =t+5 and e(1) =1. Thus $|e(0)-e(1)| = |t+4|$, and $K(2^k, 3,4)$ is not balanced.

<u>**Subcase 2.**</u> Assume c has label 1.

If t+3's 0 are label on the leave vertices x_i, then we see that e(1) =t+7 and e(0) =0. Thus $|e(0)-e(1)| = |t+7|$, and $K(2^k, 3,4)$ is not balanced.

If t+2's 0 are label on the leave vertices x_i, then

(1) if one of $\{v_1,v_2\}$ is labeled by 0 and all of $\{v_3,v_4,v_5\}$ are labeled by 1, we see that e(0) =t+6 and e(1) =1. Thus $|e(0)-e(1)| = |t+5|$, and $K(2^k, 3,4)$ is not balanced.

(2) if all of $\{v_1,v_2\}$ are labeled by 1 and one of $\{v_3,v_4,v_5\}$ is labeled by 0, we see that e(1) =t+5 and e(0) =0. Thus $|e(0)-e(1)| = |t+5|$, and $K(2^k, 3,4)$ is not balanced.

If t+1's 0 are label on the leave vertices x_i, then

(1) if all of $\{v_1,v_2\}$ are labeled by 0, we see that e(1) =t+6 and e(0) =1. Thus $|e(0)-e(1)| = |t+5|$, and $K(2^k, 3,4)$ is not balanced.

(2) if one of $\{v_1,v_2\}$ is labeled by 0 and one of $\{v_3,v_4,v_5\}$ is labeled by 0, we see that e(0) =t+4 and e(1) =0. Thus $|e(0)-e(1)| = |t+4|$, and $K(2^k, 3,4)$ is not balanced.

(3) if two of $\{v_3,v_4,v_5\}$ are labeled by 0, we see that e(1) =t+4 and e(0) =1. Thus $|e(0)-e(1)| = |t+3|$, and and $K(2^k, 3,4)$ is not balanced.

If t's 0 are label on the leave vertices x_i, then

(1) if $\{v_1,v_2\}$ are labeled by 1, we see that e(0) =3 and e(1) =t+4. Thus $|e(0)-e(1)| = |t+1|$, and $K(2^k, 3,4)$ is not balanced if $t \neq 0$, i.e. $k\neq 1$.

(2) if one of $\{v_1,v_2\}$ is labeled by 0 and one of $\{v_3,v_4,v_5\}$ is labeled by 1, we see that e(1) =t+3 and e(0) =1. Thus $|e(0)-e(1)| = |t+2|$, and $K(2^k, 3,4)$ is not balanced.

(3) if two of $\{v_3,v_4,v_5\}$ are labeled by 1, and $\{v_1,v_2\}$ are labeled by 0 we see that e(1) =t+4 and e(0) =1. Thus $|e(0)-e(1)| = |t+3|$, and $K(2^k, 3,4)$ is not balanced .

If t-1's 0 are label on the leave vertices x_i, then

(1) if all of $\{v_3,v_4,v_5\}$ are labeled by 0, and one of $\{v_1,v_2\}$ is labeled by 0 ,we see that e(1) =t+3 and e(0) =3. Thus $|e(0)-e(1)| = |t|$, and $K(2^k, 3,4)$ is not balanced if $t \neq 0,1$, i.e. $k \neq 1,3$.

(2) if all of $\{v_1,v_2\}$ are labeled by 0 and one of $\{v_3,v_4,v_5\}$ is labeled by 1, we see that e(1) =t+3 and e(0) =2. Thus $|e(0)-e(1)| = |t+1|$, and $K(2^k, 3,4)$ is not balanced.

<u>**Case 2**</u>. k is even, say k=2t, where $t \geq 1$.
Then $p(K(2^k, 3^2)) = 2t+5$. Thus v(0)= t+2 and = v(1)= t+3.

<u>**Subcase 1**</u>. Assume c has label 0.

If t+1's 0 are label on the leave vertices x_i, then we see that e(0) =t+1 and e(1) = 2. Thus $|e(0)-e(1)| = |t-1|$ and $K(2^k, 3^2)$ is not balanced if $t \neq 1,2$, i.e. $k \neq 2,4$.

If t's 0 are label on the leave vertices x_i, then we see that e(0) =t+1 and e(1) =1. Thus $|e(0)-e(1)| = t$ and $K(2^k, 3^2)$ is not balanced if $t \neq 1$, i.e. $k \neq 2$.

If t-1's 0 are label on the leave vertices x_i, then we see that e(0) =t+2 and e(1) =1. Thus $|e(0)-e(1)| = t+1$ and $K(2^k, 3^2)$ is not balanced.

If t-2's 0 are label on the leave vertices x_i, then we see that e(0) =t +2 and e(1) =0. Thus $|e(0)-e(1)| = t+2$ and $K(2^k, 3^2)$ is not balanced.

<u>**Subcase 2**</u>. Assume c has label 1.

If t+2's 0 are label on the leave vertices x_i, then we see that e(0) =0 and e(1) = t+4. Thus $|e(0)-e(1)| = t+4$ and $K(2^k, 3^2)$ is not balanced.

If t+1's 0 are label on the leave vertices x_i, then we see that e(0) =0 and e(1) = t+3. Thus $|e(0)-e(1)| = t+3$ and $K(2^k, 3^2)$ is not balanced.

If t's 0 are label on the leave vertices x_i, then either two 0's are in the same triangle or each triangle contains only one 0.

For the former case, we have e(0) = 1 and e(1) =t+3.

For the later case, we have e(0) = 0 and e(1) =t+2.

Both cases imply $|e(0)-e(1)| = t+2$ and $K(2^k, 3^2)$ is not balanced.

If t-1's 0 are label on the leave vertices x_i, then we see that e(0) =1 and e(1) = t+2. Thus $|e(0)-e(1)| = t+1$ and $K(2^k, 3^2)$ is not balanced.

If t-2's 0 are label on the leave vertices x_i, then we see that e(0) =1 and e(1) = t+2. Thus $|e(0)-e(1)| = t$ and $K(2^k, 3^2)$ is not balanced if $t \neq 1$, i.e. $k \neq 2$.□

<u>**Theorem 4.2.**</u> The windmill graph $K(2^k, 3^2,4)$ is balanced if and only if $k \leq 7$.
Proof. Table of Figure 18 shows the windmill graph $K(2^k, 3^2,4)$ is balanced if $k \leq 7$.

n	c	v_1	v_2	v_3	v_4	v_5	v_6	v_7	$\{x_1,x_2,\ldots,x_n\}$
1	0	1	1	1	1	1	0	0	0
2	0	1	1	1	1	1	0	0	0,0
3	0	1	1	1	1	1	1	0	0,0,0
4	0	1	1	1	1	1	1	0	0,0,0,0
5	0	1	1	1	1	1	1	1	0, 0,0,0,0
6	0	1	1	1	1	1	1	1	0, 0,0,0,0,0
7	0	1	1	1	1	1	1	1	0, 0,0,0,0,0,1

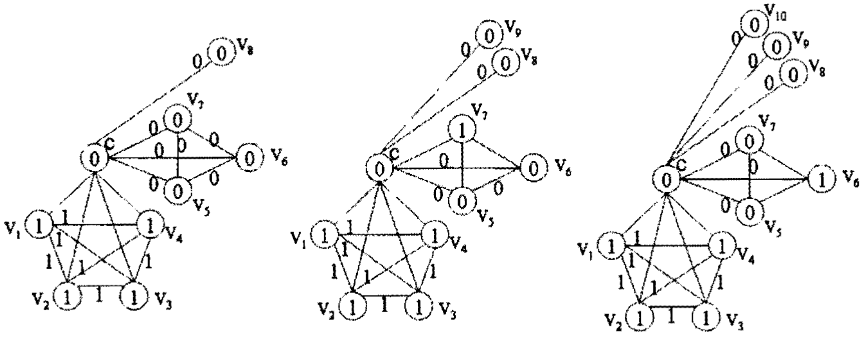
Figure 18.

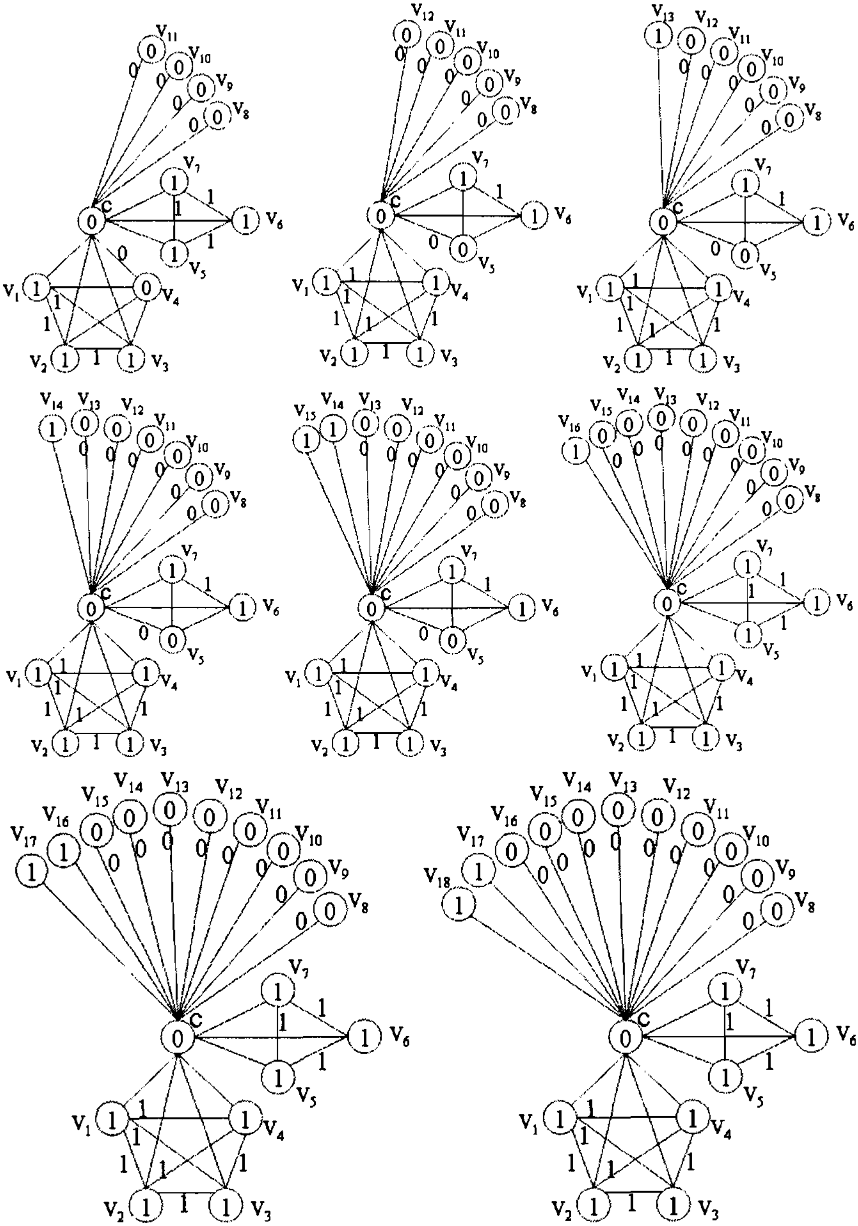
Theorem 4.3. The windmill graph $K(2^k, 3,5)$ is balanced if and only if $k \leq 12$.
Proof. Table of Figure 19 shows the windmill graph $K(2^k, 3,5)$ is balanced if $k \leq 12$.

n	c	v_1	v_2	v_3	v_4	v_5	v_6	$\{x_1,x_2,\ldots,x_n\}$
1	0	1	1	1	0	0	1	0
2	0	1	1	1	1	0	0	0,0
3	0	1	1	1	0	1	1	0,0,0
4	0	1	1	1	1	0	1	0,0,0,0
5	0	1	1	1	1	1	1	1, 0,0,0,0
6	0	1	1	1	1	1	1	0, 0,0,0,0,0
7	0	1	1	1	1	1	1	1, 0,0,0,0,0,0
8	0	1	1	1	1	1	1	1, 0,0,0,0,0,0,0
9	0	1	1	1	1	1	1	1,1, 0,0,0,0,0,0,0
10	0	1	1	1	1	1	1	1,1,1, 0,0,0,0,0,0,0
11	0	1	1	1	1	1	1	1,1,1,0,0,0,0,0,0,0,0
12	0	1	1	1	1	1	1	1,1,1,1,0,0,0,0,0,0,0,0

Figure 19.

Theorem 4.4. The windmill graph $K(2^k, 4,5)$ is balanced if and only if $k \leq 15$.





c
v1 v2 v3 v4 v5 v6 v7 v8 v9 v10 v11
v1 v2 v3 v4 v5 v6 v7 v8 v9 v10 v11 v12
v1 v2 v3 v4 v5 v6 v7 v8 v9 v10 v11 v12 v13
v1 v2 v3 v4 v5 v6 v7 v8 v9 v10 v11 v12 v13 v14
v1 v2 v3 v4 v5 v6 v7 v8 v9 v10 v11 v12 v13 v14 v15
v1 v2 v3 v4 v5 v6 v7 v8 v9 v10 v11 v12 v13 v14 v15 v16
v1 v2 v3 v4 v5 v6 v7 v8 v9 v10 v11 v12 v13 v14 v15 v16 v17
v1 v2 v3 v4 v5 v6 v7 v8 v9 v10 v11 v12 v13 v14 v15 v16 v17 v18

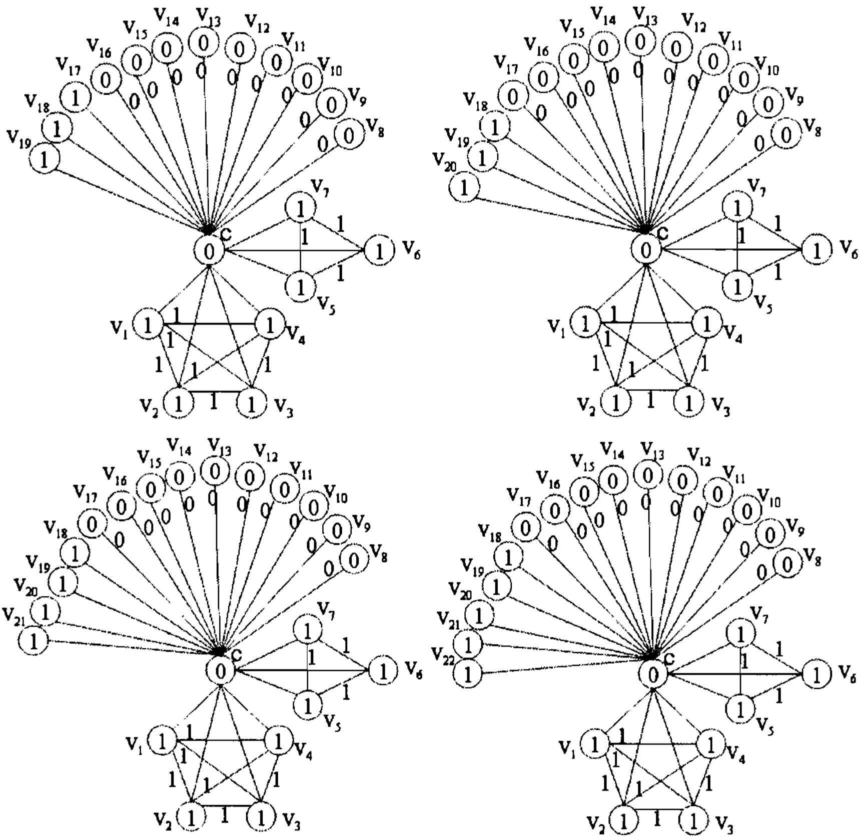


Figure 20.

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