[9] C. S. ReVelle, Solving the Roman Legions Problem, *John Hopkins Magazine* **49** (4) (1997) http://www.jhu.edu/~jhumag/0997web/roman.html

[10] C. S. ReVelle and K. E. Rosing, Defendens Imperium Romanum: A classical problem in military strategy, *American Mathematical Monthly* **107** (7) (2000) 585–594.

[11] R. R. Rubalcaba, P. J. Slater, and A. Schneider, A survey on graphs which have equal domination and closed neighborhood packing numbers, *AKCE International Journal of Graphs and Combinatorics* **3** (2) (2006) 163174.

[12] R. R. Rubalcaba, M. Walsh, Minimum fractional dominating and maximum fractional packing functions, submitted.

[13] I. Stewart, Defend the Roman Empire!, *Scientific American* **281** (6) (1999) 136–139.

On Balance Index Sets of Chain Sum and Amalgamation of Generalized Theta Graphs

Harris Kwong
Department of Mathematical Sciences
State University of New York at Fredonia
Fredonia, NY 14063, USA

Sin-Min Lee
Department of Computer Science
San Jose State University
San Jose, CA 95192, USA

Abstract

Let G be a graph with vertex set V and edge set E. Any vertex labeling $f : V \to \{0,1\}$ induces a partial edge labeling $f^* : E \to \{0,1\}$ defined by $f^*(xy) = f(x)$ if and only if $f(x) = f(y)$. Let $v_f(i) = |f^{-1}(i)|$ and $e_f(i) = |f^{*-1}(i)|$. We call f a friendly labeling if $|v_f(0) - v_f(1)| \leq 1$. The balance index set of G is defined as $BI(G) = \{|e_f(0) - e_f(1)| : f \text{ is friendly}\}$. In particular, G is said to be balanced if $BI(G)$ contains 0 or 1. In this paper, we study the balance index sets of generalized theta graphs, their chain sums and one-point unions.

Keywords and phrases: balance index set, generalized theta graphs, chain sum, amalgamation.

AMS 2000 MSC: 05C78

1 Introduction

In 1992, Lee, Liu and Tan [6] introduced a new labeling problem, which has recently enjoyed a renewed interest [1, 2, 3, 9, 10, 11]. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Any vertex labeling $f : V(G) \to \{0,1\}$ induces an edge partial labeling $f^* : E(G) \to \{0,1\}$ defined by

$$f^*(xy) = f(x) \text{ if and only if } f(x) = f(y)$$

CONGRESSUS NUMERANTIUM **187** (2007), pp. 21-32

for each edge $xy \in E(G)$. Notice that, if the two vertices incident to an edge are labeled differently, that edge will be unlabeled. This explains why f^* is a partial labeling of the edge set $E(G)$. For each $i = 0, 1$, define

$$\begin{aligned} v_f(i) &= |\{v \in V(G) : f(v) = i\}|, \\ e_f(i) &= |\{e \in E(G) : f^*(e) = i\}|. \end{aligned}$$

Following [7], We call f a ***friendly labeling*** if $|v_f(0) - v_f(1)| \le 1$. Simply put, the vertices of G are evenly divided into 0- and 1-vertices. We call G ***balanced*** if it admits a friendly labeling f such that $|e_f(0) - e_f(1)| \le 1$.

Obviously, many friendly labelings are far from balanced. To measure the closeness to being balanced, the following notion was introduced by Lee, Lee and Ng [6]. The ***balance index set*** of G, denoted $\mathrm{BI}(G)$, is defined as

$$\mathrm{BI}(G) = \{|e_f(0) - e_f(1)| : f \text{ is balanced}\}.$$

Example 1. Figure 1 displays a graph G with $\mathrm{BI}(G) = \{0, 1, 2\}$. □

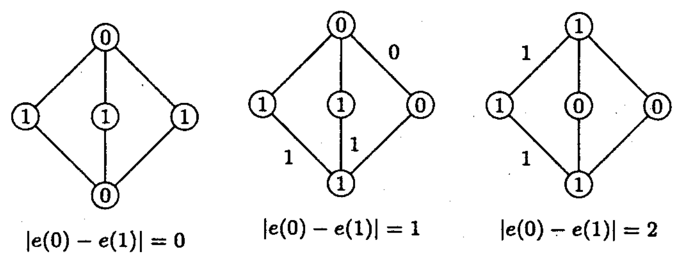

Figure 1: A graph with balance index set $\{0, 1, 2\}$.

Example 2. For a cycle C_n with vertex set $\{x_1, x_2, \ldots, x_n\}$, we denote by $C_n(t)$ the cycle with a chord x_1x_t. The balance index sets of $C_4(3)$, $C_6(4)$ and $C_6(5)$ are depicted in Figure 2. All of them equal to $\{0, 1\}$. □

In both examples, the balance index sets consist of arithmetic progressions. In general, this is not always true.

Example 3. For the graph G displayed below in Figure 3, we find $\mathrm{BI}(G) = \{0, 1, 2, 3, 4, 6\}$, in which the number 5 is missing. □

The missing term needs not be the second to the last entries. Let $H(n, m)$ denote the graph obtained from m copies of the star $\mathrm{St}(n) = K_{1,n}$

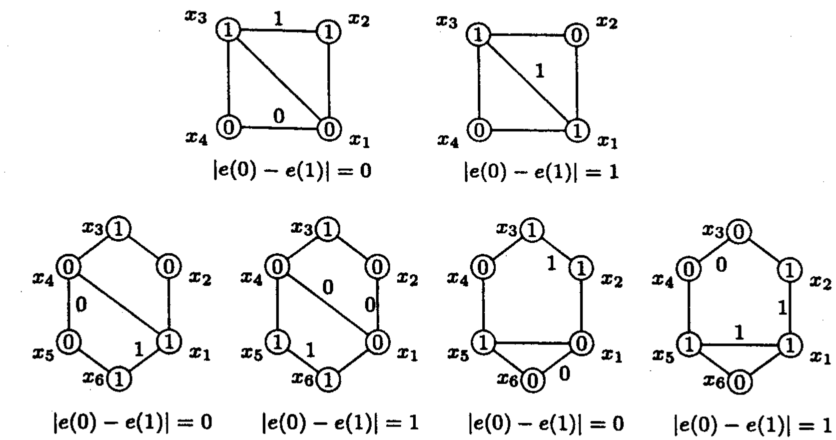

Figure 2: The balance index sets of $C_4(3)$, $C_6(4)$ and $C_6(5)$.

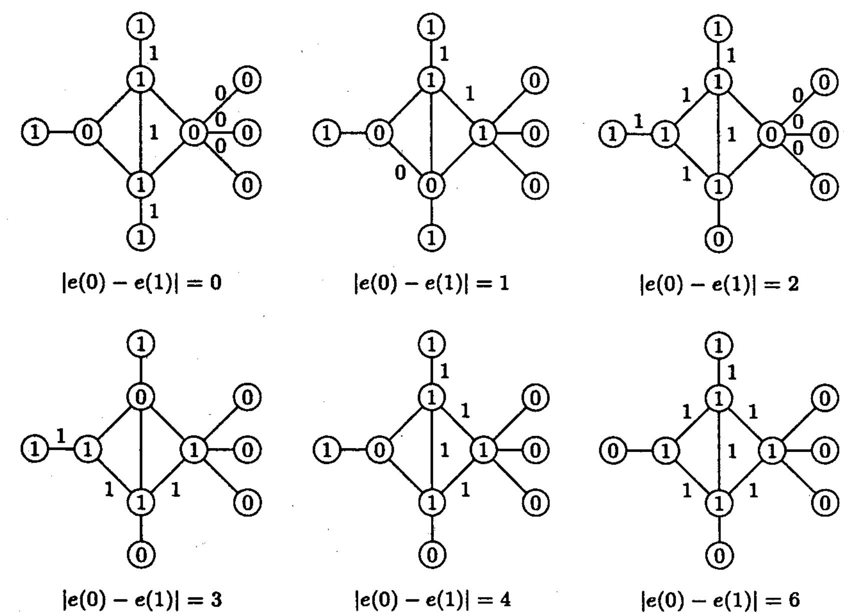

Figure 3: A graph whose balance index set is $\{0, 1, 2, 3, 4, 6\}$.

by identifying one of their pendant vertices. The authors of this paper and Sarvate found [4]

$$\mathrm{BI}(H(4,5)) = \{0,1,2,\dots,10\} - \{8\}.$$

Some balance index sets could have more than one term missing [4]:

$$\mathrm{BI}(H(4,4)) = \{1,2,\dots,8\} - \{3,6\}.$$

Sometimes, the missing entries could split into several arithmetic progressions [4]:

$$\mathrm{BI}(H(6,5)) = \{0,1,2,\dots,15\} - (\{2,7,12\} \cup \{3,8,13\}).$$

In this paper, we shall discuss the balance index sets of generalized theta graphs, their chain sums and one-point union. Other results on balance index sets can be found in [4, 5, 8, 10].

2 Generalized Theta Graphs

Take k paths of length $\ell_1, \ell_2, \dots, \ell_k$, where $k \geq 3$ and $\ell_i = 1$ for at most one i. Identify their endpoints to form a new graph. The new graph is called a ***generalized theta graph***, and is denoted $\Theta(\ell_1, \ell_2, \dots, \ell_k)$. In other words, $\Theta(\ell_1, \ell_2, \dots, \ell_k)$ consists k pairwise internally disjoint paths of length $\ell_1, \ell_2, \dots, \ell_k$ that share a pair of common endpoints. See Figure 4.

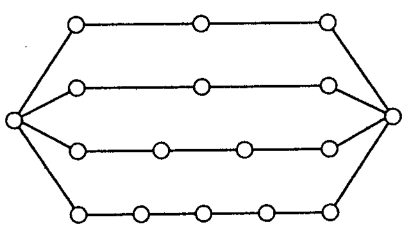

Figure 4: The generalized theta graph $\Theta(4,4,5,6)$.

When $k = 3$, the generalized theta graph is traditionally called a ***theta graph***. For brevity, when $\ell_1 = \ell_2 = \cdots = \ell_k = a$, we will write $\Theta(a^k)$. Clearly, $\Theta(2^k)$ is the complete bipartite graph $K_{2,k}$.

The ***generalized book*** $B(n_1, n_2, \dots, n_k)$ composes of k cycles of length $n_1, n_2, \dots, n_k$ that share a common edge called its ***base***. Obviously, we have $B(n_1, n_2, \dots, n_k) = \Theta(1, n_1 - 1, n_2 - 1, \dots, n_k - 1)$.

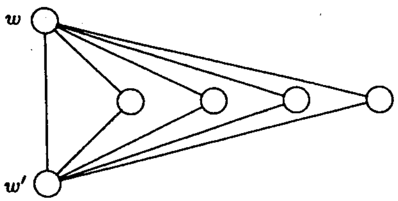

Figure 5: The generalized book $B(3,3,3,3)$.

Example 4. The generalized book $B(3,3,3,3)$ depicted in Figure 5 is also the generalized theta graphs $\Theta(1,2,2,2,2)$. □

Before we discuss the balance index set of a generalized theta graph, we first study the possible values that $e(1) - e(0)$ could attain on a single path.

Lemma 2.1 *Let w and w' be the two endpoints of a path, and let f be a vertex labeling (which needs not be friendly), then*

$$e(0) - e(1) = \begin{cases} v(0) - v(1) - 1 & \text{if } f(w) = f(w') = 0, \\ v(0) - v(1) & \text{if } f(w) \neq f(w'), \\ v(0) - v(1) + 1 & \text{if } f(w) = f(w') = 1. \end{cases}$$

Proof. First, consider $f(w) = 0$. Starting with w, assume there are c_1 consecutive 0-vertices, followed by d_1 consecutive 1-vertices, then c_2 consecutive 0-vertices, then d_2 consecutive 1-vertices, and so forth, as indicated below:

$$\underbrace{0\dots0}_{c_1}\underbrace{1\dots1}_{d_1}\underbrace{0\dots0}_{c_2}\underbrace{1\dots1}_{d_2}\dots$$

The number of blocks of such 0- and 1-vertices is either $2b$ or $2b+1$, depending on whether $f(w') = 1$ or $f(w') = 0$, respectively. Note that $b \geq 1$. It is easy to verify that

$$e(1) = \sum_{i \geq 1} (d_i - 1) = \left(\sum_{i \geq 1} d_i \right) - b = v(1) - b.$$

In a similar fashion, we find

$$e(0) = \sum_{i \geq 1} (c_i - 1) = \begin{cases} v(0) - b - 1 & \text{if } f(w') = 0, \\ v(0) - b & \text{if } f(w') = 1. \end{cases}$$

They immediately prove our claim in the case of $f(w) = 0$. The case of $f(w) = 1$ is handled in a similar manner. □

Lemma 2.2 *Let w and w' be the two points that identify the endpoints of the paths in $\Theta(\ell_1, \ell_2, \ldots, \ell_k)$. Then for any vertex labeling f,*

$$e(0) - e(1) = \begin{cases} v(0) - v(1) + k - 2 & \text{if } f(w) = f(w') = 0, \\ v(0) - v(1) & \text{if } f(w) \neq f(w'), \\ v(0) - v(1) - k - 2 & \text{if } f(w) = f(w') = 1. \end{cases}$$

Proof. Let $v_i(0)$, $v_i(1)$, $e_i(0)$ and $e_i(1)$ be the restrictions of the respective numbers on each path. First consider $f(w) = 0$. If $f(w') = 0$, then $e_i(0) - e_i(1) = v_i(0) - v_i(1) - 1$ for each i. Therefore, over the entire generalized theta graph,

$$e(0) - e(1) = \sum_{i \geq 1} v_i(0) - \sum_{i \geq 1} v_i(1) - k.$$

It is clear that $\sum_{i \geq 1} v_i(1) = v(1)$. But, because of over-counting,

$$\sum_{i \geq 1} v_i(0) = v(0) + 2(k - 1).$$

It follows that $e(0) - e(1) = v(0) - v(1) + k - 2$. Next, assume $f(w') = 1$. We have

$$\begin{aligned} e(0) - e(1) &= \sum_{i \geq 1} v_i(0) - \sum_{i \geq 1} v_i(1) \\ &= [v(0) + k - 1] - [v(1) + k - 1] \\ &= v(0) - v(1). \end{aligned}$$

The result for $f(w) = f(w') = 1$ is obtained in a similar fashion. □

Theorem 2.3 *For any $\Theta(\ell_1, \ell_2, \ldots, \ell_k)$, let $\ell = \sum_{i \geq 1} \ell_i$, then*

$$BI(\Theta(\ell_1, \ell_2, \ldots, \ell_k)) = \begin{cases} \{0, k-2\} & \text{if } \ell - k \text{ is even,} \\ \{1, k-3, k-1\} & \text{if } \ell - k \text{ is odd.} \end{cases}$$

Proof. Let w and w' be the two points that identify the endpoints of the k paths. The generalized theta graph has $\ell - k + 2$ vertices, hence

$$v(0) - v(1) = \begin{cases} 0 & \text{if } \ell - k \text{ is even,} \\ \pm 1 & \text{if } \ell - k \text{ is odd.} \end{cases}$$

Note that if all the vertices are replaced by their complements, a friendly labeling remains friendly, and $|e(0) - e(1)|$ remains unchanged. Hence we may assume w is a 0-vertex. That leaves w' either a 0-vertex or an 1-vertex.

If $f(w') = 0$, according to Lemma 2.2,

$$\begin{aligned} e(0) - e(1) &= v(0) - v(1) + k - 2 \\ &= \begin{cases} k - 2 & \text{if } \ell - k \text{ is even,} \\ k - 3 \text{ or } k - 1 & \text{if } \ell - k \text{ is odd.} \end{cases} \end{aligned}$$

If $f(w') = 1$, we find

$$\begin{aligned} e(0) - e(1) &= v(0) - v(1) \\ &= \begin{cases} 0 & \text{if } \ell - k \text{ is even,} \\ \pm 1 & \text{if } \ell - k \text{ is odd.} \end{cases} \end{aligned}$$

It is clear that all these vertex labelings are attainable. The proof is now completed by combining the two cases. □

Notice that the values in the balance index set of a generalized Θ-graph depends only on the number of paths it contains.

Corollary 2.4 *For any $B(n_1, n_2, \ldots, n_k)$, let $n = \sum_{i \geq 1} n_i$, then*

$$BI(B(n_1, n_2, \ldots, n_k)) = \begin{cases} \{0, k-1\} & \text{if } n \text{ is even,} \\ \{1, k-2, k\} & \text{if } n \text{ is odd.} \end{cases}$$

Corollary 2.5 *Let $k \geq 2$, then*

$$BI(B(a^k)) = \begin{cases} \{0, k-1\} & \text{if } ak \text{ is even,} \\ \{1, k-2, k\} & \text{if } ak \text{ is odd.} \end{cases}$$

Corollary 2.6 *For $n_1, n_2 \geq 3$,*

$$BI(B(n_1, n_2)) = \begin{cases} \{0, 1\} & \text{if } n_1 + n_2 \text{ is even,} \\ \{0, 1, 2\} & \text{if } n_1 + n_2 \text{ is odd.} \end{cases}$$

Corollary 2.7 *For $n \geq 3$,*

$$BI(B(3^k, n)) = \begin{cases} \{0, k\} & \text{if } n + k \text{ is even,} \\ \{1, k-1, k+1\} & \text{if } n + k \text{ is odd.} \end{cases}$$

3 Two-Chain Sums of Generalized Θ-Graphs

Let $\Theta(\ell_1, \ell_2, \ldots, \ell_k)$ be a generalized theta graph with s_1 and s_2 at both ends, and $\Theta(m_1, m_2, \ldots, m_r)$ be another generalized theta graph with u_1 and u_2 at both ends. We can construct a new graph by forming their disjoint union and identifying the vertices s_2 with u_1. We call the resulting graph their ***two-chain sum***, and denote it $\Theta(\ell_1, \ell_2, \ldots, \ell_k) \# \Theta(m_1, m_2, \ldots, m_r)$. The two-chain sum $\Theta(4, 4, 5, 6) \# \Theta(3, 4, 6)$ is displayed in Figure 6.

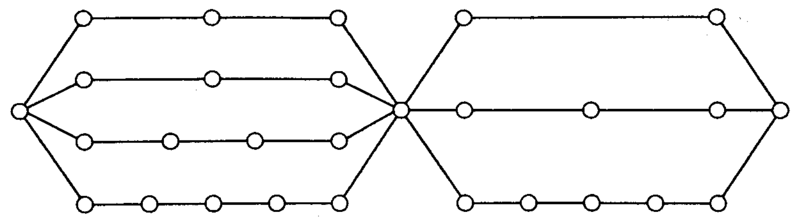

Figure 6: The two-chain sum $\Theta(4,4,5,6)\#\Theta(3,4,6)$.

Theorem 3.1 *Let $G = \Theta(\ell_1, \ell_2, \ldots, \ell_k)\#\Theta(m_1, m_2, \ldots, m_r)$. Define $N = \ell + m - k - r$, where $\ell = \sum_{i\geq 1} \ell_i$ and $m = \sum_{i\geq 1} m_i$. Then*

$$BI(G) = \begin{cases} \{0,2,k-2,k,r-2,r,k+r-4,k+r-2\} & \text{if } N \text{ is even,} \\ \{1,k-1,r-1,k+r-3\} & \text{if } N \text{ is odd.} \end{cases}$$

Proof. Let w be the vertex that is shared by the two generalized theta graphs, and let w' and w'' be the vertex on the other end of $\Theta(\ell_1, \ell_2, \ldots, \ell_k)$ and $\Theta(m_1, m_2, \ldots, m_r)$ respectively. Given any friendly labeling f, we may assume $f(w) = 0$. We need to consider four cases.

If $f(w') = f(w'') = 0$, the restriction of f on $\Theta(\ell_1, \ell_2, \ldots, \ell_k)$ gives, according to Lemma 2.2,

$$e'(0) - e'(1) = v'(0) - v'(1) + k - 2,$$

where e' and v' indicate the value of e and v within $\Theta(\ell_1, \ell_2, \ldots, \ell_k)$. Likewise, the restriction of f on $\Theta(m_1, m_2 \ldots, m_r)$ yields

$$e''(0) - e''(1) = v''(0) - v''(1) + r - 2.$$

It is clear that $e'(0)+e''(0) = e(0)$, $e'(1)+e''(1) = e(1)$, and $v'(1)+v''(1) = v(1)$. However, due to over-counting of $f(w)$, we find $v'(0)+v''(0) = v(0)+1$. We conclude that

$$e(0) - e(1) = v(0) - v(1) + k + r - 3.$$

The two-chain sum graph has $N + 3$ vertices, therefore

$$e(0) - e(1) = \begin{cases} k+r-4, k+r-2 & \text{if } N \text{ is even,} \\ k+r-3 & \text{if } N \text{ is odd.} \end{cases}$$

In a similar manner, if $f(w') = 0$ and $f(w'') = 1$, we find

$$e(0) - e(1) = \begin{cases} k-2, k & \text{if } N \text{ is even,} \\ k-1 & \text{if } N \text{ is odd.} \end{cases}$$

If $f(w') = 1$ and $f(w'') = 0$, we have

$$e(0) - e(1) = \begin{cases} r-2, r & \text{if } N \text{ is even,} \\ r-1 & \text{if } N \text{ is odd.} \end{cases}$$

If $f(w') = f(w'') = 1$, we have

$$e(0) - e(1) = \begin{cases} 0, 2 & \text{if } N \text{ is even,} \\ 1 & \text{if } N \text{ is odd.} \end{cases}$$

Note that all these vertex labelings are attainable. Combining the values from all four cases yields the desired result immediately. □

Not too surprisingly, Theorem 3.1 asserts that the values in the balance index set of a 2-chain sum of two generalized Θ-graphs depend only on the number of paths in the two generalized Θ-graphs. In particular, we obtain the following.

Corollary 3.2 *Let $G = \Theta(\ell^k)\#\Theta(m^r)$, where $\ell, m \geq 2$ and $k, r \geq 3$. Then*

$$BI(G) = \begin{cases} \{0,2,k-2,k,r-2,r,k+r-4,k+r-2\} & \text{if } N \text{ is even,} \\ \{1,k-1,r-1,k+r-3\} & \text{if } N \text{ is odd,} \end{cases}$$

where $N = (\ell-1)k + (m-1)r$.

Example 5. We find

$$\mathrm{BI}(\Theta(2^3)\#\Theta(2^k)) = \begin{cases} \{0,1,2,3,k-2,k-1,k,k+1\} & \text{if } k \text{ is odd,} \\ \{1,2,k-1,k\} & \text{if } k \text{ is even,} \end{cases}$$

for any integer $k \geq 3$. □

4 *t*-Chain Sums of Generalized Θ-Graphs

The notion of a two-chain sum can be naturally extended to multiple copies of generalized theta graphs. A ***t*-chain sum** is obtained by joining t generalized theta graphs in a "head-to-tail" fashion. Although we can apply the same technique to derive its balance index set, as one may expect, the result is unpleasantly complicated. Nonetheless, we determine, in this section, the balance index set of the 3-chain sum $\Theta(\ell^k)\#\Theta(m^r)\#\Theta(n^s)$.

Let w, w', w'', w''' be the four vertices at the ends of the three individual generalized theta graphs, starting from $\Theta(\ell^k)$ and ending with $\Theta(n^s)$. Let f be a friendly labeling of the 3-chain sum. Restricted to each individual generalized theta graph, we use Lemma 2.2 to compute δ such that

$$e(0) - e(1) = v(0) - v(1) + \delta.$$

The values of the three δ's (one from each generalized theta graph), which we shall denote δ', δ'' and δ''', are tabulated below. Their sum, along with the adjustment ϵ from the over-counting of 0- and 1-vertices, gives us the value of Δ such that, over the entire 3-chain sum,

$$e(0) - e(1) = v(0) - v(1) + \Delta.$$

As usual, we may assume $f(w) = 0$.

$f(w)$	$f(w')$	$f(w'')$	$f(w''')$	δ'	δ''	δ'''	ϵ	Δ
0	0	0	0	$k-2$	$r-2$	$s-2$	+2	$k+r+s-4$
0	0	0	1	$k-2$	$r-2$	0	+2	$k+r-2$
0	0	1	0	$k-2$	0	0	0	$k-2$
0	0	1	1	$k-2$	0	$2-s$	0	$k-s$
0	1	0	0	0	0	$s-2$	0	$s-2$
0	1	0	1	0	0	0	0	0
0	1	1	0	0	$2-r$	0	−2	$-r$
0	1	1	1	0	$2-r$	$2-s$	−2	$-(r+s-2)$

Since the number of vertices in the 3-chain sum is $k(\ell-1)+r(m-1)+s(n-1)+4$, and the vertex labelings are obviously attainable, we obtain the following result.

Theorem 4.1 *Let G denote $\Theta(\ell_1,\ldots,\ell_k)\#\Theta(m_1,\ldots,m_r)\#\Theta(n_1,\ldots,n_s)$, where $\ell,m,n \geq 2$ and $k,r,s \geq 3$. Define*

$$\ell = \sum_{i\geq 1} \ell_i, \qquad m = \sum_{i\geq 1} m_i, \qquad n = \sum_{i\geq 1} n_i,$$

and

$$\begin{aligned} N &= k(\ell-1)+r(m-1)+s(n-1), \\ T &= \{0, k-2, r, s-2, k-s, k+r-2, r+s-2, k+r+s-4\}, \\ S &= \{x \pm 1 \mid x \in T\}. \end{aligned}$$

Then

$$BI(G) = \begin{cases} \{|z| : z \in T\} & \text{if } N \text{ is even,} \\ \{|z| : z \in S\} & \text{if } N \text{ is odd.} \end{cases}$$

5 Amalgamation of Generalized Θ-Graphs

The two-chain sum of two generalized theta graphs can be regarded as their amalgamation or one-point union. This prompts us to study the amalgamation of t generalized theta graphs.

For the sake of brevity, for each i where $1 \leq i \leq t$, denote by Θ_i the generalized theta graph that consists of k_i internally disjoint paths of length ℓ_{ij}, where $1 \leq j \leq k_i$. Denote by $\mathrm{Amal}(\Theta_1, \Theta_2, \ldots, \Theta_t)$ the amalgamation at one of the two ends of each Θ_i. The amalgamation has $\ell - k + t + 1$ vertices, where $\ell = \sum_{i,j} \ell_{ij}$ and $k = \sum_i k_i$. Let w be the center, and w_i the other end of Θ_i. Given any friendly labeling f, we may assume $f(w) = 0$. Using the same argument we used above, we find

$$e(0) - e(1) = v(0) - v(1) + t - 1 + \sum_{i=1}^{t} (1 - f(w_i))(k_i - 2),$$

from which we can easily deduce the following result.

Theorem 5.1 *Let $\ell = \sum_{i,j} \ell_{ij}$ and $k = \sum_i k_i$. Then*

$$BI(Amal(\Theta_1, \Theta_2, \ldots, \Theta_t)) = \begin{cases} \{|z| : z \in S\} & \text{if } \ell+k+t \text{ is even,} \\ \{|z| : z \in T\} & \text{if } \ell+k+t \text{ is odd,} \end{cases}$$

where

$$T = \left\{ t - 1 + \sum_{i=1}^{t} \epsilon_i (k_i - 2) \;\middle|\; \epsilon_i = 0, 1 \text{ for each } i \right\},$$

and $S = \{x \pm 1 \mid x \in T\}$.

In particular, when each Θ_i is $\Theta(\ell^k)$, we obtain the following result.

Corollary 5.2 *For $\ell \geq 2$, $k \geq 3$, and $t \geq 1$, we have*

$$BI(Amal(\Theta(\ell^k)^t)) = \begin{cases} \{|z| : z \in S\} & \text{if } tk(\ell-1)+t \text{ is even,} \\ \{|z| : z \in T\} & \text{if } tk(\ell-1)+t \text{ is odd,} \end{cases}$$

where

$$\begin{aligned} T &= \{i(k-2)+t-1 \mid 0 \leq i \leq t\}, \\ S &= \{i(k-2)+t-2, i(k-2)+t \mid 0 \leq i \leq t\}. \end{aligned}$$

References

[1] Y.S. Ho, S.M. Lee, H.K. Ng and Y.H. Wen, On balancedness of some families of trees, manuscript.

[2] R.Y. Kim, S.M. Lee and H.K. Ng, On balancedness of some families of graphs, manuscript.

[3] M.C. Kong, S.M. Lee, E. Seah and A.S. Tang, A complete characterization of balanced graph, manuscript.

[4] H. Kwong, S.M. Lee and D.G. Sarvate, On balance index sets of one-point unions of graphs, manuscript.

[5] A.N.T. Lee, S.M. Lee and H.K. Ng, On balance index sets of graphs, *J. Combin. Math. Combin. Comput.*, to appear.

[6] S.M. Lee, A. Liu and S.K. Tan, On balanced graphs, *Congr. Numer.* **87** (1992), 59–64.

[7] S.M. Lee and H.K. Ng, On friendly index sets of bipartite graphs, *Ars Combin.*, to appear.

[8] S.M. Lee, Y.C. Wang and Y.H. Wen, On the balance index sets of the $(p, p+1)$-graphs, *J. Combin. Math. Combin. Comput.*, to appear.

[9] M.A. Seoud and A.E.I. Abdel Maqsoud, On cordial and balanced labelings of graphs, *J. Egyptian Math. Soc.* **7** (1999) 127–135.

[10] D.H. Zhang, Y.S. Ho, S.M. Lee and Y.H. Wen, On balance index sets of trees with diameter at most four, manuscript.

[11] D.H. Zhang, S.M. Lee and L. Wen, On balancedness of the galaxies with at most four stars, manuscript.

Self-Referential Derivation

Okechukwu Chidume, LaShundra Griswold,
Dean Hoffman, and Peter Johnson*
Department of Mathematics and Statistics
Auburn University, AL 36849
johnspd@auburn.edu

Key words and phrases: sequence, multiset, Martin Gardner, Clifford Pickover

Abstract

Self-referential derivation is an "operation" on finite sequences dreamed up by the mathematical recreationalist Clifford Pickover; it may be the offspring of a bona fide operation on finite sequences introduced by Martin Gardner. Here we find all possible self-referential derivatives of all finite sequences of distinct non-negative integers.

1 Introduction

A sequence $(b_0, \ldots, b_{n-1}) = \bar{b}$ is a *self-referential*, or *Pickover*, derivative of a sequence $(a_0, \ldots, a_{n-1}) = \bar{a}$ if, for each $i \in \{0, \ldots, n-1\}$, b_i is the number of times a_i appears in $\bar{b}$ – that is, $b_i = |\{j | b_j = a_i\}|$. For example, $(1, 2, 2)$ is its own Pickover derivative, but it has 4 others: $(2, 1, 1), (1, 0, 0), (0, 2, 2)$, and $(0, 0, 0)$. Of these 4, only $(2, 1, 1)$ has a Pickover derivative – $(0, 0, 0)$. It is interesting to note that $(0), (0, 1)$, and $(0, 1, 2)$ have no Pickover derivatives, but $(0, 1, 2, 3)$ has two, $(1, 2, 1, 0)$ and $(2, 0, 2, 0)$. Of these two, only $(1, 2, 1, 0)$ has Pickover derivatives, $(0, 0, 0, 3)$ and $(0, 2, 0, 2)$, and of these two, only $(0, 0, 0, 3)$ has a Pickover derivative, but it is the ancestor of an infinite succession of Pickover derivatives, $(1, 1, 1, 0)$ and $(0, 0, 0, 3)$ alternating forever.

Self-referential derivation was introduced (but not by that or any other name) by Clifford Pickover in a last-page column on mathematical recreations in *Discover* magazine in 1997 [2]. He suggested a number of problems, the chief one being to find the Pickover derivatives of the sequences $(0, 1, \ldots, n-1)$, for all $n \geq 1$.

*This research was supported by the National Science Foundation under Grant no. 0353723

CONGRESSUS NUMERANTIUM **187** (2007), pp. 33-39