

On Group–magic Eulerian Graphs

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Abstract

Let A be an abelian group. We call a graph $G = (V, E)$ A –**magic** if there exists a labeling $f : E(G) \rightarrow A^*$ such that the induced vertex set labeling $f^+ : V(G) \rightarrow A$, defined by $f^+(v) = \Sigma f(u, v)$ where $(u, v) \in E(G)$, is a constant map. In this paper, we present some algebraic properties of A –magic graphs. Using them, various results are obtained for group–magic eulerian graphs.

1 Introduction.

Let G be a connected (multi)graph, with no loops. For any abelian group A (written additively), let $A^* = A - \{0\}$. A function $f : E(G) \rightarrow A^*$ is called a *labeling* of G . Any such labeling induces a map $f^+ : V(G) \rightarrow A$, defined by $f^+(v) = \Sigma f(u, v)$, where $(u, v) \in E(G)$. If there exists a labeling f whose induced map on $V(G)$ is a constant map, we say that f is an A –*magic labeling* and that G is an A –*magic graph*. The *integer–magic spectrum* of a graph G is the set $\{k : G \text{ is } Z_k\text{–magic and } k \geq 2\}$.

In this article, we will use the following notation. Let $[G, A]$ denote the class of distinct A –magic labelings of G . Note that G is A –magic if and only if $[G, A] \neq \emptyset$. For any ring R with unity, $U(R)$ denotes the multiplicative group of units in R .

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Z -magic graphs were considered by Stanley [10,11], where he pointed out that the theory of magic labelings could be studied in the general context of linear homogeneous diophantine equations. Doob [1,2,3] has studied A -magic graphs and Z_k -magic were investigated in [5,6,7].

Within the mathematical literature, various definitions of magic graphs have been introduced. The original concept of an A -magic graph is due to J. Sedlacek [8,9], who defined it to be a graph with real-valued edge labeling such that (i) distinct edges have distinct nonnegative labels, and (ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices. Previously, Kotzig and Rosa [4] had introduced yet another definition of a magic graph. Over the years, there has been great research interest in graph labeling problems. The interested reader is directed to Wallis' [12] recent monograph on magic graphs.

2 A necessary condition for G to be Z_3 -magic.

It is straight-forward to determine a necessary and sufficient condition for G to be Z_2 -magic. Clearly, G is Z_2 -magic if and only if every vertex of G is of the same parity. It seems that finding a similar condition for G to be Z_3 -magic is much more difficult. We establish the following result:

Theorem 1. *Let G be Z_3 -magic, with p vertices and q edges. Let $f \in [G, Z_3]$ induce the constant label x on the vertices of G , and $|E_i|$ denote the number of edges labeled i . Then, $px \equiv q + |E_1|, \pmod{3}$.*

Proof. With any Z_3 -magic labeling f of G , we can associate a multigraph \widehat{G} with G . \widehat{G} is formed by replacing every edge in G which was labeled 2, with two edges labeled 1. Note that \widehat{G} is a Z_3 -magic multi-graph with p vertices and $2|E_2| + |E_1|$ edges. In any (p, q) -graph, we have that $\sum \deg(v_i) = 2q$. Since all of the edges in \widehat{G} are labeled 1, this implies that $px \equiv 4|E_2| + 2|E_1|, \pmod{3}$. From this, we see that $px \equiv |E_2| + |E_1| + |E_1|, \pmod{3}$. Thus, $px \equiv q + |E_1|, \pmod{3}$. \square

Several remarks should be made with regard to Theorem 1. First, note that with similar calculations, one can easily derive an analogous result in terms of $|E_2|$ (ie: $px + q + |E_2| \equiv 0, \pmod{3}$). Also, Theorem 1 might be used to reduce the number of calculations performed, when trying to find a Z_3 -magic labeling of G via computer search. In addition, similar necessary conditions can be established for graphs G to be Z_k -magic.

3 Algebraic properties of A -magic graphs.

After examining a few examples by hand, the observant reader will note a sort of duality appearing in Theorem 1. The next two results give a reason as to why this occurs.

Theorem 2. *Let A be a non-trivial denumerable abelian group, underlying some ring R with unity. If $d \in U(A)$ and $f \in [G, A]$, then $df \in [G, A]$.*

Proof. Suppose that f induces a constant label x on all of the vertices of G . Consider an arbitrary vertex v and let $|E_i|$ denote the number of edges labeled a_i , which are adjacent to v . Then, $x = \Sigma(a_i|E_i|)$; where a_i varies through all the elements of A^* . Let us examine what effect df has on the labeling of v . By multiplying every edge adjacent to v by d , we get the following relationship: $dx = d\Sigma(a_i|E_i|)$. The new induced labeling on v is dx . Also, since $d \in U(A)$, each edge adjacent to v in this new labeling is not equal to 0_A . Thus, df is an A -magic labeling of G . \square

The following result is an immediate consequence of Theorem 2.

Corollary 1. *If $d \in U(Z_k)$ and $f \in [G, Z_k]$, then $df \in [G, Z_k]$.*

Proof. Let $A = Z_n$, the group of integers, modulo n . Now, apply Theorem 2. \square

It should be noted that in Theorem 2 and Corollary 1, f and df might yield the same group-magic labeling on G . Also, a natural question to ask is the following: If G is an A -magic graph, does there exist some labeling of G for which all other possible A -magic labelings arise, by applying this group action? In general, the answer is no. Consider the following two labelings for a Z_9 -magic graph G :

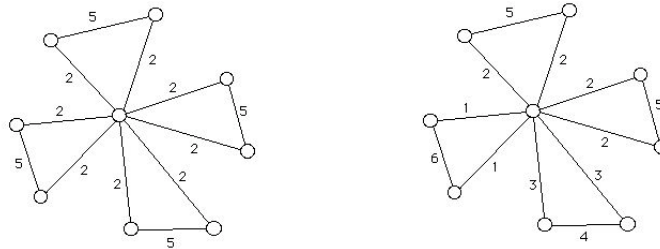


Figure 1.

Let $f \in [G, Z_9]$ and $d_1, d_2 \in U(Z_9)$ where $d_i f$ gives the i^{th} labeling. By multiplying each edge in the first labeling by $d_2 d_1^{-1}$, we obtain the second labeling. However, this is impossible.

We wish to continue to develop an algebraic framework from which group-magic graphs can be analyzed. Some of the following results will give us additional tools for studying A -magic eulerian graphs.

Theorem 3. *Let A_1 be an abelian group which contains a subgroup isomorphic to A_2 . If graph G is A_2 -magic, then G is A_1 -magic.*

Proof. Let $H \leq A_1$. Suppose that $f \in [G, A_2]$ and that $\phi : A_2 \rightarrow H$ is a group isomorphism. Now, let f induce a constant label x on all of the vertices of G . Consider an arbitrary vertex v and let $|E_i|$ denote the number of edges labeled a_i , which are adjacent to v . Then, $x = \Sigma(a_i|E_i)$; where a_i varies through all the elements of A_2^* . Now, apply ϕ to the edges which are adjacent to v . Under this new labeling, we get the following relationship: $\phi(x) = \phi[\Sigma(a_i|E_i)] = \Sigma\phi(a_i|E_i)$. Since $a_i \neq 0_{A_2}$ and ϕ is a group isomorphism, no edge is labeled 0_{A_1} . The new induced labeling on v is $\phi(x)$. Hence, we have an A_1 -magic labeling of G . \square

Although the next result is an immediate corollary of Theorem 3, for the sake of clarity, a detailed proof has been given.

Corollary 2. *Let G be a Z_k -magic graph, with $k|n$. Then, G is a Z_n -magic graph.*

Proof. Suppose that we have a Z_k -magic labeling on G . Let x be the constant label on the vertices of G and suppose that $kd = n$. Now, consider an arbitrary vertex v and let $|E_i|$ denote the number of edges labeled i , which are adjacent to v . Then, $x \equiv \Sigma(i|E_i), \text{ mod } k$; where i varies from 1 to $k-1$. By multiplying every edge adjacent to v by d , we get the following relationship: $dx \equiv d\Sigma(i|E_i), \text{ mod } kd$ and hence $dx \equiv d\Sigma(i|E_i), \text{ mod } n$. The new induced labeling on v is dx . Also, since $1 \leq i \leq k-1$ and $d \neq 0$, we have that $0 < di < n$. In particular, di is not congruent to 0, mod n . Thus, in this new labeling, no edge is labeled 0. Since v was taken to be an arbitrary vertex, we have shown that G is Z_n -magic. \square

The reader should observe that the converse of Corollary 2 is not true. (ie. If G is Z_n -magic, with $k|n$, it does not follow that G is Z_k -magic.) For example, let G be the eulerian graph consisting of a C_4 block and a C_3 block and sharing one common vertex ($p = 6, q = 7$). Now, G is Z_2 -magic. By Corollary 2, G is Z_6 -magic. However, it is straight-forward to verify that G is not Z_3 -magic.

Also, Corollary 2 allows us to obtain information about the integer-magic spectrum of G . For example, if G is Z_p -magic for all primes p , then G is Z_n -magic for all $n \geq 2$.

4 Results on eulerian graphs.

There are still many open questions with regard to the characterization of A -magic eulerian graphs. In this section, an assortment of results is given.

Corollary 3. *Every eulerian graph G is Z_k -magic, for k even.*

Proof. This follows immediately from Corollary 2. \square

Corollary 4. *Let A be an abelian group containing an element of order 2. Then, every eulerian graph G is A -magic.*

Proof. Suppose that x is an involution in A . Label every edge of G with x . Then, every vertex of G has an induced labeling of 0. Hence, G is A -magic. \square

Note that Corollary 4 also follows from Theorem 3.

Theorem 4. *Let A be any non-trivial abelian group. Then, every eulerian graph G with an even number of edges must be A -magic.*

Proof. Suppose that $a \in A$, with $a \neq 0$. Let $e_1 e_2 e_3 \cdots e_{2n}$ be an eulerian circuit, starting and ending at vertex v . The following labeling scheme will give an A -magic labeling of G :

$$f(e_i) = \begin{cases} a, & \text{if } i \text{ is odd.} \\ -a, & \text{if } i \text{ is even.} \end{cases}$$

Note that every vertex has an induced labeling of 0. \square

We have already seen an example of an eulerian graph with an odd number of edges, which is not A -magic (e.g. G , consisting of a C_4 block and a C_3 block and sharing one common vertex). In contrast to this, there are also eulerian graphs with an odd number of edges, which are A -magic. For example, any odd cycle is A -magic.

Let us now focus our attention on eulerian graphs with an odd number of edges. First, we begin with a definition.

Definition 1. Given a connected graph $G = (V, E)$, let $T(G)$ denote the graph which is obtained from G by adding a disjoint uv -path of length 2 between every adjacent pair of vertices u, v in $V(G)$.

Note that $T(G)$ has $|V(G)| + |E(G)|$ vertices and $|E(G)| + 2|E(G)|$ edges. Also, it is straight-forward to show that $T(G)$ is eulerian. We investigate the following question: For which G with $|E(G)|$ odd, is $T(G)$ A -magic for all non-trivial, finite abelian groups A ?

Theorem 5. *$T(P_{2k})$ is A -magic, where P_{2k} is a path of order $2k$ and A is any non-trivial, finite abelian group.*

Proof. Let $a \in A$, with $a \neq 0$. The following diagram gives an A -magic labeling of $T(P_{2k})$.

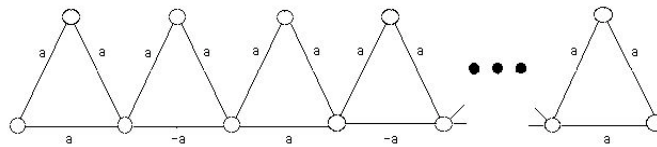


Figure 2.

\square

Theorem 6. $T(K_{1,2n+1})$ is A -magic, for all non-trivial, finite abelian groups A .

Proof. First, we show that $T(K_{1,2n+1})$ is Z_p -magic, for all prime p . Since $T(K_{1,2n+1})$ is eulerian (and thus is Z_2 -magic), let $p \geq 3$. Furthermore, let v denote the vertex of degree $2(2n+1)$ and E_v be the set of edges incident to v in $T(K_{1,2n+1})$. There are two cases to consider: (i). $2(2n+1) \equiv 1, \pmod{p}$ and (ii). $2(2n+1) \not\equiv 1, \pmod{p}$.

(i). $2(2n+1) \equiv 1, \pmod{p}$. From E_v , label $2(2n+1) - 2$ edges with 1 and the remaining two edges, with $p - 1$. Now, v has an induced labeling of $2(2n+1) - 2 + 2(p-1) \equiv p - 3, \pmod{p}$. We label the remaining edges of $T(K_{1,2n+1})$ in the following manner: If the edge is adjacent to two edges labeled 1, then label it $p - 4, \pmod{p}$; otherwise, label the edge $p - 2, \pmod{p}$. This yields a Z_p -magic labeling of $T(K_{1,2n+1})$.

(ii). $2(2n+1) \not\equiv 1, \pmod{p}$. Label every edge in E_v with 1. Note that v has an induced labeling of $2(2n+1) \equiv 4n+2, \pmod{p}$. We label the remaining edges of $T(K_{1,2n+1})$ with $4n+1, \pmod{p}$. Since $4n+2 \not\equiv 1, \pmod{p}$, we have that $4n+1 \not\equiv 0, \pmod{p}$. Hence, we have a Z_p -magic labeling of $T(K_{1,2n+1})$.

Therefore, $T(K_{1,2n+1})$ is Z_p -magic, for all prime p . Now, every finite abelian group A can be written as a direct sum of cyclic groups, each of order a power of a prime. Also, every finite p -group has Z_p as a subgroup. Hence, by Theorem 3, $T(K_{1,2n+1})$ is A -magic. □

Here is another construction which yields eulerian graphs with an odd number of edges.

Definition 2. A graph H is *homeomorphic from* G if either H is isomorphic to G or H is isomorphic to a graph obtained by subdividing some sequence of edges of G .

Definition 3. A *cycle-snake* is any graph which is homeomorphic from $T(P_k)$.

The reader will note that by making subdivisions on the edges of $T(P_k)$, for $k \geq 2$, many eulerian graphs having an odd number of edges can be created. We will show that certain types of cycle-snakes are A -magic, for all non-trivial, abelian groups A . Before we do that, the following lemma is needed.

Lemma 1. Let graph G have an A -magic labeling with a vertex-induced label x . Furthermore, let $a \in A^*$, where $2a = x$, and $E_a(G)$ denote the edges of G which are labeled a . Then, any graph obtained from G by subdividing edges in $E_a(G)$ is A -magic.

Proof. Since edges in $E_a(G)$ are subdivided, they are replaced with paths of length 2 in the new graph. Label the edges of the paths with a . The new graph obtained will be A -magic, having the same vertex-induced label as G . □

Theorem 7. Let $E_a[T(P_{2k})]$ denote the set of edges corresponding to the A -magic labeling found in Figure 2. Then, any cycle-snake obtained from $T(P_{2k})$ by subdividing edges in $E_a[T(P_{2k})]$ is A -magic.

Proof. This follows immediately from Theorem 5 and Lemma 1. □

Not all cycle-snakes are A -magic. For example, it is straight-forward to show that the graph consisting of a C_4 block and a C_3 block and sharing one common vertex, has integer-magic spectrum equaling $2N$.

Up to this point, the integer-magic spectrum of the eulerian graphs that we have come across has either been $2N$ or $N - \{1\}$. This might lead the reader to believe that this is the case for all eulerian graphs. However, there are eulerian graphs, whose integer-magic spectrum is neither $2N$ nor $N - \{1\}$. For example, the following diagram gives a Z_7 -magic labeling of an eulerian graph. It is straight-forward to show that this graph is not Z_3 -magic.

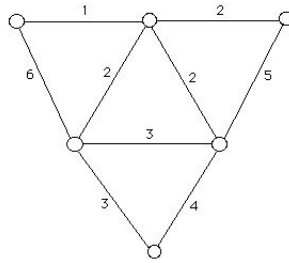


Figure 3.

5 Directions for further research.

Open Problem 1. *Characterize the A -magic eulerian graphs with an odd number of edges.*

Open Problem 2. *Find necessary and sufficient conditions for a graph G to be Z_3 -magic.*

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