On $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic Graphs

Yihui Wen
Department of Mathematics
Suzhou Science and Technology College
Suzhou, Jiangsu 215009, Peoples Republic of China

Sin-Min Lee
Department of Computer Science
San Jose State University
San Jose, CA 95192, USA

Hsin-Hao Su
Department of Mathematics
Stonehill College
Easton, MA 02357, USA

Abstract
For any abelian group $\mathbf{A}$ written additively, we denote $\mathbf{A}^* = \mathbf{A} - \{0\}$. Let $G$ be a graph with a vertex set $V(G)$ and an edge set $E(G)$. A mapping $l : E(G) \to \mathbf{A}^*$ is called a labeling. Given a labeling on the edge set of $G$, we can induced a vertex set labeling $l^+ : V(G) \to \mathbf{A}$ as follow:

$$l^+(v) = \sum \{l(u,v) \mid (u,v) \in E(G)\}.$$

A graph $G$ is known as $\mathbf{A}$-magic if there is a labeling $l : E(G) \to \mathbf{A}^*$ such that for each vertex $v$, the sum of the labels of the edges incident with $v$ are all equal to the same constant. In this paper, we investigate graphs which are $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic and not $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic.

1 Introduction

For any abelian group $\mathbf{A}$ written additively, we denote $\mathbf{A}^* = \mathbf{A} - \{0\}$. Let $G$ be a graph with a vertex set $V(G)$ and an edge set $E(G)$. A mapping $l : E(G) \to \mathbf{A}^*$ is called a labeling. Given a labeling on the edge set of $G$, we can induced a vertex set labeling $l^+ : V(G) \to \mathbf{A}$ as follow:

$$l^+(v) = \sum \{l(u,v) \mid (u,v) \in E(G)\}.$$
A graph $G$ is known as A-magic if there is a labeling $l : E(G) \rightarrow A^*$ such that for each vertex $v$, the sum of the labels of the edges incident with $v$ are all equal to the same constant; i.e., $l^+(v) = c$ for some fixed $c$ in $A$. We call $<G, l>$ an A-magic graph. In general, a graph $G$ may admit more than one labelings to become an A-magic graph.

We denote the class of all graphs (either simple or multiple graphs) by $\text{Gph}$. The class of all abelian groups is named by $\text{Ab}$. For each $A$ in $\text{Ab}$, we denote the class of all A-magic graphs by $\text{AGph}$.

When $A = \mathbb{Z}$, the Z-magic graphs were considered in Stanley [20]; he pointed out that the theory of magic labelings can be put into the more general context of linear homogeneous diophantine equations (see [21]).

When the group is $\mathbb{Z}_k$, we shall refer to the $\mathbb{Z}_k$-magic graph as $k$-magic. Graphs which are $k$-magic had been studied in [13].

Doob [2, 3, 4] also considered A-magic graphs where $A$ is an abelian group. Given a graph $G$, the problem of deciding whether $G$ admits a magic labeling is equivalent to the problem of deciding whether a set of linear homogeneous Diophantine equation has a solution (see [17]). At present, given an abelian group, no general efficient algorithm is known for finding magic labelings for general graphs.

The original concept of A-magic graph is due to J. Sedlacek [18, 19], who defined it to be a graph with real-valued edge labeling such that (i) distinct edges have distinct nonnegative labels, and, (ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

In this paper we use $\mathbb{N}$ to denote the set of natural numbers, $\{1, 2, 3, \ldots\}$, and for each $k > 0$, we write the set $\{kx \mid x \in \mathbb{N}\}$ by $k\mathbb{N}$ and $\{k+x \mid x \in \mathbb{N}\}$ by $k + \mathbb{N}$. We also define the graph $G$ with a magic labeling $l : E(G) \rightarrow \mathbb{N}$ as $\mathbb{N}$-magic. It is well-known that a graph $G$ is $\mathbb{N}$-magic if and only if each edge of $G$ is contained in a 1-factor (a perfect matching) or a $\{1, 2\}$-factor. The Z-magic is weaker than N-magic. Figure 1 shows that a graph which is Z-magic but not N-magic.

![Figure 1: Z-magic but not N-magic](image)

The Klein-four group $V_4$ is the direct sum $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Its multiplication
looks like

<table>
<thead>
<tr>
<th>⊕</th>
<th>(0, 0)</th>
<th>(0, 1)</th>
<th>(1, 0)</th>
<th>(1, 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>(0, 0)</td>
<td>(0, 1)</td>
<td>(1, 0)</td>
<td>(1, 1)</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>(0, 1)</td>
<td>(0, 0)</td>
<td>(1, 1)</td>
<td>(1, 0)</td>
</tr>
<tr>
<td>(1, 0)</td>
<td>(1, 0)</td>
<td>(1, 1)</td>
<td>(0, 0)</td>
<td>(0, 1)</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>(1, 1)</td>
<td>(1, 0)</td>
<td>(0, 1)</td>
<td>(0, 0)</td>
</tr>
</tbody>
</table>

It is easy to see that \( V_4 \) is abelian and \( a + b + c = 0 \) in \( V_4 \). For simplicity, we denote \((0, 0), (0, 1), (1, 0)\) and \((1, 1)\) by \(0, a, b\) and \(c\), respectively. Thus, \( V_4 \) is the group \( \{0, a, b, c\} \) with the operation \( + \) as:

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>c</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

There are only a few papers related to \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-magic graphs. In [10], Saba, Salehi, Sun and the second author were the pioneers to consider this problem. In this paper, we investigate some more graphs which are \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-magic, or are not \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-magic.

2 Graphs which are both \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \text{-V} \) magic and \( \mathbb{Z}_2 \)-magic

Consider the wheel graph \( W_{n+1} = N_1 + C_n \) where \( V(W_{n+1}) = \{c_0\} \cup \{c_1, \ldots, c_n\} \) and \( E(W_{n+1}) = \{(c_0, c_i) \mid i = 1, \ldots, n\} \cup E(C_n) \). Note here in order to have a cycle, we require that \( n \geq 3 \).

**Theorem 2.1.** For an integer \( n \geq 3 \), the wheel graph \( W_{n+1} = N_1 + C_n \) is both \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-magic and \( \mathbb{Z}_2 \)-magic if \( n \) is odd.

**Proof.** In a wheel \( W_{n+1} \), there are \( n \) edges adjacent to its center \( c_0 \).

Since \( n \geq 3 \), there are at least 3 edges adjacent to \( c_0 \). We label \((c_0, c_1)\) by \( b \), \((c_0, c_2)\) by \( c \) and all other edges adjacent to \( c_0 \) by \( a \). Next, we label the edge \((c_1, c_2)\) by \( a \) and all other edges in \( C_n \), by \( b \) and \( c \), alternatively, starting from \((c_2, c_3)\) (see Figure 2).

Thus, the sum of the vertex \( c_0 \) is \( a + b + c + (n - 3)a \). Since the characteristic of \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) is 2 and \( n \) is odd, the sum is 0. For all other vertices on \( C_n \), they are all order 3 whose three edges are labeled by \( a, b \) and \( c \), respectively. Therefore, the sum is also 0.

Note that all the vertices on \( C_n \) part of a wheel graph are order 3. The order of \( c_0 \) is \( n \), which is also odd. Simply label all edges 1 and then the wheel graph \( W_{n+1} \) becomes \( \mathbb{Z}_2 \)-magic. □
In Section 3, we see that $W_{n+1}$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic, but not $\mathbb{Z}_2$-magic if $n$ is even.

**Definition 1.** A graph $G$ is called an odd graph if its degree sequence $<d_1, d_2, \ldots, d_n>$ has the property that $d_i$ is odd for all $i \geq 1$.

**Theorem 2.2.** All odd graphs are both $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic and $\mathbb{Z}_2$-magic.

**Proof.** If $G$ is an odd graph, we see that if we label all edges of $G$ by 1 then $G$ is $\mathbb{Z}_2$-magic with sum 1.

Pick any non-zero element $x$ of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and label every edge of $G$ by $x$. Since the order of each vertex is odd, the sum is $x$. Thus, it is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic with sum $x$. \qed

**Example 1.** Figure 3 shows that $K_4$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic with sum 0 and $\mathbb{Z}_2$-magic with sum 1.

- Figure 3: A $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic labeling and a $\mathbb{Z}_2$-magic labeling of $K_4$

**Definition 2.** A graph $K_4(a_1, a_2, a_3)$ where $a_i \geq 1$ for all $i = 1, 2, 3$ is a graph which is formed by a $K_4$ where $V(K_n) = \{v_0, v_1, v_2, v_3\}$, and for each $v_i$, where $i = 1, 2, 3$, there exists $a_i$ pendant edges.

**Example 2.** Figure 4 demonstrates $K_4(1, 2, 3)$.
Figure 4: $K_4(1, 2, 3)$

**Corollary 2.3.** A $K_4(a_1, a_2, a_3)$ where $a_i \geq 1$ for all $i = 1, 2, 3$ is both $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic and $\mathbb{Z}_2$-magic if $a_i$ is even for all $i = 1, 2, 3$.

**Proof.** With the condition $a_i$ is even and greater or equal to 1 for all $i = 1, 2, 3$, by definition, it is easy to see that a $K_4(a_1, a_2, a_3)$ is an odd graph. \qed

# 3 Graphs which are $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic but not $\mathbb{Z}_2$-magic

**Theorem 3.1.** The wheel graph $W_{n+1} = N_1 + C_n$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic, but not $\mathbb{Z}_2$-magic if $n$ is even.

**Proof.** In a wheel $W_{n+1}$, there are $n$ edges adjacent to its center $c_0$.

We label all edges adjacent to $c_0$ by $a$, and all edges on $C_n$ by $b$ and $c$, alternatively (see Figure 5).

Figure 5: A $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic labeling of $W_5$
Thus, the sum of the vertex $c_0$ is $n a$. Since the characteristic of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is $2$ and $n$ is even, the sum $n a$ is $0$. For all other vertices on $C_n$, they are all order $3$ whose three edges are labeled by $a$, $b$ and $c$, respectively. Therefore, the sum is also $0$.

Note that all the vertices on $C_n$ part of a wheel graph are order $3$. The order of $c_0$ is $n$ which is even. Thus, a wheel $W_{n+1}$ cannot be $\mathbb{Z}_2$-magic if $n$ is even. \hfill \Box

Consider the graph $d(G) = < 1, 2, 3 >$ in Figure 6. We name the pendant vertex $u$ and the other vertices and edges as follow:

![Figure 6: $d(G)$]

Figure 7 shows that $d(G)$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic.

![Figure 7: A $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic labeling of $d(G)$]

But, since there the order sequence of $d(G)$ is $\{1, 2, 2, 3\}$, it is not $\mathbb{Z}_2$-magic.

From now on, we rename $d(g)$ as $(G, u)$. The amalgamation of copies of $(G, u)$ is formed by gluing the pendant vertex $u$ of each copy into a common vertex. We denote the amalgamation of $n$ copies of $(G, u)$ by $\text{Amal}(n, (G, u))$. Thus, $(G, u)$ is $\text{Amal}(1, (G, u))$. Figure 8 demonstrates $\text{Amal}(2, (G, u))$ and $\text{Amal}(3, (G, u))$.

Note that $\text{Amal}(n, (G, u))$ is not $\mathbb{Z}_2$-magic for all $n$ since there are always order $2$ and $3$ vertices.

**Theorem 3.2.** The $\text{Amal}(n, (G, u))$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic but not $\mathbb{Z}_2$-magic if $n$ is odd.
**Proof.** Figure 7 shows that \( \text{Amal}(1, (G, u)) \) is \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-magic. For \( n \geq 3 \) and odd, we label each copy of \((G, u)\) as shown in Figure 7. It is easy to see that since \( n \) is odd, the sum of the vertex \( u \) is \( a \). Thus, we have a \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-magic labeling with sum \( a \). 

Let \( C_{2n} \) be a cycle with a vertex set \( V(C_{2n}) = \{c_1, \cdots, c_n, c_1^*, \cdots, c_n^*\} \) and \( \Sigma \) be a permutation of \( \{1, 2, \cdots, n\} \). We construct a cubic graph \( C_{2n}(\Sigma) \) as follows:

\[
V(C_{2n}(\Sigma)) = V(C_{2n})
\]

and

\[
E(C_{2n}(\Sigma)) = E(C_{2n}) \cup \{(c_i, c_{\Sigma(i)}^*) \mid i = 1, 2, \cdots, n\}.
\]

Next, let \( \xi : E(C_{2n}) \to \mathbb{N} \) be a mapping. For any \( E \) in \( E(C_{2n}) \), if \( \xi(E) = n_E \) and \( n_E \) is not equal to 0, then we subdivide the edge \( E \) by inserting \( n_E \) new vertices into the edge \( E \). If \( \xi(E) = 0 \), we do not add any vertex in \( E \).

We obtain a subdivision graph of \( C_{2n}(\Sigma) \), and we denote this graph by \( \text{Sub}(C_{2n}(\Sigma), \xi) \).

**Theorem 3.3.** For an integer \( n \geq 3 \), let \( \Sigma \) be any permutation of \( \{1, 2, \cdots, n\} \), and \( \xi \) be any mapping from \( E(C_{2n}) \to \mathbb{N} \). The subdivision graph \( \text{Sub}(C_{2n}(\Sigma), \xi) \) is \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-magic. It is \( \mathbb{Z}_2 \)-magic if and only if \( \xi \) is a zero mapping.

**Proof.** First, we label all the edges of \( C_{2n} \) by the sequence \( a, c, a, c, \cdots \). In the \( \text{Sub}(C_{2n}(\Sigma), \xi) \), for any edge \( E \) in \( C_{2n} \), if \( \xi(E) \) is not zero, then we label all subdivided edges by the same character \( a \) or \( c \) depending on the label of the edge \( E \). Next, we label all edges \( (c_i, c_{\Sigma(i)}^*) \) by \( b \).

Since \( C_{2n} \) has even number of edges, we start by labeling \( a \) and end by labeling \( c \). Also, every vertex \( C_{2n} \) is connected by a permutation edge \( (c_i, c_{\Sigma(i)}^*) \) for some \( i \). Thus, every vertex of \( C_{2n} \) has three edges which are labeled \( a \), \( c \), and \( b \). Therefore, the sum is \( 0 \).
For all added vertices in the edge $E$, the two connected edges are either labeled $a$ or $c$. Thus, the sum is also 0. This proves that Sub$(C_{2n}(\Sigma), \xi)$ is $Z_2 \oplus Z_2$-magic.

If $\xi$ is a zero mapping, then every vertex in Sub$(C_{2n}(\Sigma), \xi)$ is of order 3. By Theorem 2.2, it is $Z_2$-magic. Conversely, if $\xi$ is not a zero mapping, then we have added vertices in some of the edges $E$. These vertices must be order 2. But, all vertices of $C_{2n}$ are order 3. Thus, it is not $Z_2$-magic. 

**Example 3.** Figure 9 demonstrates a Sub$(C_{2n}(\Sigma), \xi)$ with its $Z_2 \oplus Z_2$-magic labeling.

![Figure 9: Sub$(C_{2n}(\Sigma), \xi)$ with its $Z_2 \oplus Z_2$-magic labeling](image)

For convenience, from now on, we name the vertices of a path $P_n$ by $v_1, v_2, \ldots, v_n$ and the edges $(v_i, v_{i+1})$ by $e_i$. We also call the vertices $v_2, \ldots, v_{n-1}$ **inside vertices** if $n \geq 3$.

**Definition 3.** For an integer $n \geq 3$, a graph Comb$(n)$ is a graph formed by a path $P_n$ and, for each inside vertex $v_i$, where $i = 2, 3, \ldots, n - 1$, there exists a pendant edge with a vertex. We call these pendant vertices by $v_i^*$. (See Figure 10.)

**Definition 4.** Let $P_n$ be the path within a graph Comb$(n)$. Let $F : E(P_n) \to \mathbb{N}$ be a mapping. A graph Village$(n, F)$ is a graph which is formed by a Comb$(n)$ and by gluing a path $P_{F(e_i)+2}$ to $v_i^*$ and $v_{i+1}^*$ for all $i = 2, \ldots, n - 2$ or to $v_i$ and $v_{i+1}^*$ for $i = 1$ or $n - 1$. (See Figure 10.)

**Example 4.** Figure 10 shows that the Comb$(7)$, and $F : P_7 \to \mathbb{N}$ with $F(e_1) = 0, F(e_2) = 3, F(e_3) = 2, F(e_4) = 1, F(e_5) = 3, F(e_6) = 2$. The resulting graph Village$(7, F)$ is $Z_2 \oplus Z_2$-magic with sum 0.
Theorem 3.4. For an integer \( n \geq 3 \), and any mapping \( F : E(P_n) \rightarrow \mathbb{N} \), the graph \( \text{Village}(n, F) \) is \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-magic, but not \( \mathbb{Z}_2 \)-magic.

Proof. First, we label all edges of \( P_n \) by the sequence \( \mathbf{a}, \mathbf{c}, \mathbf{a}, \mathbf{c}, \ldots \) and all the edges \((v_i, v_i^*)\) where \( i = 2, 3, \ldots, n - 1 \) by \( \mathbf{b} \). Next, we label all edges of newly added paths \( P_{F(e_i)+2} \) by the same character \( \mathbf{a} \) or \( \mathbf{c} \) as the edge \( e_i \) where \( i = 1, 2, \ldots, n - 1 \).

Two end vertices of \( P_n \) have only two edges labeled both \( \mathbf{a} \) or \( \mathbf{c} \). Thus, the sum is 0. Also, all the inside vertices are of order 3 with edges labeled \( \mathbf{a}, \mathbf{c} \) and \( \mathbf{b} \). Therefore, the sum is again 0.

For every vertex \( v_i^* \) where \( i = 2, 3, \ldots, n - 1 \), it has three edges. One is \((v_i, v_i^*)\) and the other two are from two adjacent newly added paths. Therefore, these three edges are labeled \( \mathbf{b}, \mathbf{a} \) and \( \mathbf{c} \). So, the sum is 0.

For all inside edges of the newly added paths, by construction, they are all of order 2 and labeled by the same character, either \( \mathbf{a} \) or \( \mathbf{c} \). This completes the proof that \( \text{Village}(n, F) \) is \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-magic.

Obviously, the vertices in the newly added paths are order 2. But, all the inside vertices of \( P_n \) are order 3. Thus, it is not \( \mathbb{Z}_2 \)-magic. \( \square \)

Definition 5. For an integer \( n \geq 2 \), let \( P_n \) and \( P_n^* \) be two paths of length \( n \) where their vertices are named by \( v_1, v_2, \ldots, v_n \) and \( v_1^*, v_2^*, \ldots, v_n^* \), respectively. A graph \( \text{Village}(n) \) is a graph formed by \( P_n \) and \( P_n^* \) with extra vertices \( r_1, r_2, \ldots, r_{n-1} \) and extra edges \((v_i, v_i^*)\) for all \( i = 1, \ldots, n \) and \((r_i, v_i^*)\) and \((r_i, v_{i+1}^*)\) for all \( i = 1, 2, \ldots, n - 1 \). (See Figure 11.)

Example 5. Figure 11 shows the graphs \( \text{Village}(n) \) where \( n = 2, 3, 4, 5 \) with their \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-magic labelings.
Figure 11: $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic labelings of Villages

**Theorem 3.5.** For an integer $n \geq 2$, the graph Village$(n)$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic, but not $\mathbb{Z}_2$-magic.

**Proof.** First, we label all edges of $P_n$ by the sequence $b, c, b, c, \ldots$ and all the edges of $P_n^*$ by the sequence $c, b, c, b, \ldots$. Next, we label $(v_1, v_1^*)$ by $c$ and $(v_n, v_n^*)$ by $b$. All other edges are labeled by $a$.

It is easy to see that the two edges adjacent to the vertex $r_i$ are both labeled $a$. Thus, the sum is 0.

For every vertex $v_i$ where $i = 2, 3, \ldots, n-1$, since we label the edges of $P_n$ by the sequence $b, c, b, c, \ldots$ and the edge $(v_i, v_i^*)$ by $a$, the sum is 0.

For every vertex $v_i^*$ where $i = 2, 3, \ldots, n-1$, it has five edges. Three of them are labeled $b$, $a$ and $c$. The other two edges are both connected to the vertex $r_i$. Thus, they are both labeled $a$. Therefore, the sum is 0.

Finally, by construction, the two edges of $v_1$ are labeled $c$ and the two edges of $v_n$ are labeled $b$. Therefore, the sum is 0. Also, by construction, the three edges of $v_1^*$, $(v_1, v_1^*)$, $(v_1^*, v_2^*)$, and $(r_1, v_1^*)$ are labeled by $c$, $b$ and $a$, respectively. The sum is also 0. Similarly, the sum of $v_n^*$ is again 0. This completes the proof that Village$(n)$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic.

Obviously, $v_1$ is order 2. But, $v_2$ is order 3. Thus, it is not $\mathbb{Z}_2$-magic. $\square$

We define the planar graph Pagoda$(n)$ of level $n$ inductively as follows:
Definition 6.

1. Pagoda(1) is a graph which combines one edge of 3-cycle $C_3$ with one edge side of 4-cycle $C_4$ as in Figure 12.

![Figure 12: Pagoda(1)](image)

2. Pagoda(2) is a graph which combines the bottom edge of Pagoda(1) with one edge of 4-cycle $C_4$.

3. Pagoda$(n)$ is a graph which combines the bottom edge of Pagoda$(n - 1)$ with one edge of 4-cycle $C_4$. Figure 13 demonstrates Pagoda(3).

![Figure 13: Pagoda(3)](image)

Theorem 3.6. Pagoda$(n)$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic, but not $\mathbb{Z}_2$-magic for all $n \geq 1$.

Proof. We label all side edges by the sequence $c, a, c, a, \ldots$, two top edges by $a$ and all inside edges by $b$.

Next, if $n$ is odd, then label the bottom edge by $c$; or if $n$ is even, then label the bottom edge by $a$.

It is easy to see that, for all but the bottom two side vertices, the three edges adjacent to it are labeled $a$, $b$, and $c$, respectively. Thus, the sum is 0.

The top vertex is order 2 with two edges both labeled $a$. Thus, the sum is 0.

The bottom two side vertices are order 2 with two edges which are both either labeled $c$ if $n$ is odd or labeled $a$ if $n$ is even. Thus, the sum is 0.

Therefore, Pagoda$(n)$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic.
All but the bottom two side vertices are order 3. Any other vertices are order 2. Thus, Pagoda(n) is not $\mathbb{Z}_2$-magic.

Example 6. Figure 14 shows the graphs Pagoda(3) and Pagoda(4) with their $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic labelings.

![Z_2 \oplus Z_2-magic labelings of Pagoda(3) and Pagoda(4)](image)

Figure 14: $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic labelings of Pagoda(3) and Pagoda(4)

We define inductively the planar graph Mongolian Tent $MT(n)$ with $n$ rooms as follows:

Definition 7.

1. Mongolian Tent (1), or $MT(1)$, is Pagoda(1).

2. $MT(2)$ is a graph which combines vertices and edges of the right hand side of a $MT(1)$ with vertices and edges of the left hand side of another $MT(1)$. Figure 15 demonstrates $MT(2)$.

![MT(2)](image)

Figure 15: $MT(2)$

3. $MT(n)$ is a graph which combines vertices and edges of the right hand side of a $MT(n - 1)$ with vertices and edges of the left hand side of
another MT(1), the corresponding vertices and edges are similar to 2.

In other words, MT(1) is a Mongolian Tent with one room, MT(2) is a Mongolian Tent with two rooms, and MT(n) is a Mongolian Tent with n rooms.

Definition 8.

1. A roof edge of MT(n) is an edge adjacent with the top vertex of MT(n).
2. A ceiling edge of MT(n) is an edge in the top horizontal segment of MT(n).
3. A floor edge of MT(n) is an edge in the bottom horizontal segment of MT(n).
4. A wall edge of MT(n) is a vertical edge of MT(n) between the top horizontal segment and the bottom horizontal segment.

In Figure 15, MT(2) has three roof edges, two ceiling edges, three wall edges, and two floor edges.

Theorem 3.7. Mongolian Tent, MT(n), is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic, but not $\mathbb{Z}_2$-magic for all $n \geq 1$.

Proof. It is easy to see the MT(n) has at least one vertex of order 2 and at least one vertex of order 3. Thus, MT(n) is not $\mathbb{Z}_2$-magic for all n.

If $n$ is odd, then we label all roof edges by $a$, ceiling edges by $b$, inside wall edges by $a$, and two sided wall edges by $c$. We also label the floor edges by the sequence $c, b, c, b, \ldots$.

Since $n$ is odd, there are $n + 1$ (even) roof edges adjacent with the top vertex. They are all labeled $a$. Thus, the sum is $(n + 1)a = 0$.

The inside middle vertices are of order 4 with two vertical edges labeled $a$ and two horizontal edges labeled $b$. Thus, the sum is 0.

The left and right middle vertices are of order 3. The three edges adjacent to it are labeled $a$, $b$, and $c$, respectively. Thus, the sum is 0. Also, all inside bottom vertices are similar.

Finally, the left and right bottom vertices are of order 2. The two edges adjacent to it are both labeled $c$. Thus, the sum is 0.

Therefore, if $n$ is odd, MT(n) is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic.

If $n$ is even, then we label all inside edges by $a$, left roof edge by $b$, right roof edge by $c$, left sided wall edge by $c$ and right sided wall edges by $b$. We also label the floor edges by the sequence $c, b, c, b, \ldots$.
Since $n$ is even, there are $n + 1$ (odd) roof edges adjacent with the top vertex. All inside edges are labeled $a$. Also, the left roof edge is labeled $b$, and the right roof edge is labeled $c$. Thus, the sum is $(n - 1)a + b + c = 0$. The inside middle vertices are of order 4 with four edges labeled $a$. Thus, the sum is 0.

The left and right middle vertices are of order 3. The three edges adjacent to it are labeled $a$, $b$, and $c$, respectively. Thus, the sum is 0. Also, all inside bottom vertices are similar.

Finally, the left and right bottom vertices are of order 2. The two edges adjacent to it are both labeled either $c$ or $b$. Thus, the sum is 0.

Therefore, if $n$ is even, $\text{MT}(n)$ is $Z_2 \oplus Z_2$-magic.

Example 7. Figure 16 shows the graphs $\text{MT}(3)$ and $\text{MT}(4)$ with their $Z_2 \oplus Z_2$-magic labelings.

![Figure 16: $Z_2 \oplus Z_2$-magic labelings of MT(3) and MT(4)](image)

4 Graphs which are not $Z_2 \oplus Z_2$-magic but $Z_k$-magic

From Theorem 3.2, we know that $\text{Amal}(n, (G, u))$ is $Z_2 \oplus Z_2$-magic if $n$ is odd.

Theorem 4.1. The $\text{Amal}(n, (G, u))$ is not $Z_2 \oplus Z_2$-magic but $Z_3$-magic if $n$ is even.

Proof. Assume that there is a $Z_2 \oplus Z_2$-magic labeling on $\text{Amal}(n, (G, u))$. Consider a copy of $\text{Amal}(1, (G, u))$. Since the sum of $u_1$ and $u_2$ are the same, say $x$, we have $e_1 + e_3 = x = e_2 + e_3$. This implies that $e_1$ and $e_2$ must be labeled by the same characters. Moreover, the sum of $u_0$ must be also $x$. Thus, $x = e_0 + e_1 + e_2 = e_0$ since $e_1$ equals $e_2$. But, we have even edges adjacent to $u$ which all must be labeled $x$. Therefore, the sum of $u$
must be 0. It forces \( e_0 \) to be 0 which is not allowed in a magic labeling. This is a contradiction.

Figure 17 shows that \( \text{Amal}(2, (G, u)) \) is \( \mathbb{Z}_3 \)-magic.

\[
\begin{array}{c}
\circ \ 1 \\
\circ \ 1 \\
\circ \ 1 \\
\circ \ -1 \\
\circ \ -1 \\
\circ \ 1 \\
\end{array}
\]

\[\text{Figure 17: A } \mathbb{Z}_3 \text{-magic of } \text{Amal}(2, (G, u))\]

For \( n \geq 2 \) and even, we can decompose \( \text{Amal}(n, (G, u)) \) into \( \frac{n}{2} \) copies of \( \text{Amal}(2, (G, u)) \) by coupling two copies of \( \text{Amal}(1, (G, u)) \) together. Simply label each copy of \( \text{Amal}(2, (G, u)) \) as shown in Figure 17. It is easy to see that since \( n \) is even, the sum of the vertex \( u \) is 0. Thus, we have a \( \mathbb{Z}_3 \)-magic labeling with sum 0. \( \square \)

5 Womb Graphs

**Definition 9.** For an integer \( n \geq 3 \), a womb \( \mu(n; a_1, \ldots, a_n) \) is a unicyclic graph which is formed by a cycle \( C_n \) where \( V(C_n) = \{v_1, v_2, \ldots, v_n\} \) and for each \( v_i \), there exist \( a_i \) pendant edges.

**Theorem 5.1.** For an odd integer \( n \geq 3 \), a graph \( \mu(n; 1, 0, \ldots, 0) \) is \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-magic but not \( \mathbb{Z}_2 \)-magic.

**Proof.** We label the cycle part of \( \mu(n; 1, 0, \ldots, 0) \) by \( b, c, b, c, \ldots \) and the pendant by \( a \). (See Figure 18.) Since \( n \) is odd, the labeling sequence of the cycle starts by \( b \) and ends also by \( b \). It is easy to check that it is a \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-magic labeling.

The vertices of the cycle part are order 2 except one vertex which is order 3. Therefore, it is not \( \mathbb{Z}_2 \)-magic. \( \square \)

**Example 8.** Figure 18 shows that \( \mu(5; 1, 0, \ldots, 0) \) is \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-magic but not \( \mathbb{Z}_2 \)-magic.
Theorem 5.2. A womb graph $\mu(n; a_1, \ldots, a_n)$ is both $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic and $\mathbb{Z}_2$-magic if $a_i$ is odd and greater or equal to 1 for all $i = 1, 2, \ldots, n$.

Proof. With the conditions $a_i$ is odd and greater or equal to 1 for all $i = 1, 2, \ldots, n$, by definition, it is easy to see that a womb is an odd graph. The result follows by Theorem 3.1. \qed

Example 9. Figure 19 shows that $\mu(3; 1, 3, 5)$ is both $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic and $\mathbb{Z}_2$-magic by letting $x$ to be any non-zero element in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_2$.

Figure 19: Caption for $\mu(3; 1, 3, 5)$

Theorem 5.3. For any integer $n \geq 3$ and $a_1, \ldots, a_n$ are not all zero integers, the womb graph $\mu(n; a_1, \ldots, a_n)$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic if and only if the number of the vertices in the cycle with even number of pendants is even (i.e. the number of even numbers of $a_1, \ldots, a_n$ is even.)

Proof. Since $a_1, \ldots, a_n$ are not all zero, we have at least one pendant. Without loss of generality, we can assume that the edge of this pendant is labeled $a$. Therefore, to be a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-magic graph, all the vertices must have the sum $a$. This quickly implies that all the pendants must be labeled $a$.

For a vertex $v_0$ of the cycle with odd number of pendants, the sum of all edges is $a$. Thus, to keep the sum remains $a$, the two edges on the cycle which adjacent to $v_0$ must be labeled by the same characters.
For a vertex \( v \) of the cycle with even number of pendants, the sum of all edges is 0. Thus, to keep the sum remains a, the two edges on the cycle which adjacent to \( v \) must be labeled by \( b \) and \( c \).

Assume that the number of the vertices in the cycle with even number of pendants is even. Note that if this number is 0, then, by Theorem 5.2, it is \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-magic.

If we have at least two vertices in the cycle with even number of pendants, then let us walk through every vertex in the cycle starting with a vertex with even number of pendants. No matter how many vertices with odd number of pendants between two vertices even number of pendants, we can label all the edges adjacent to them by the same character. As long as we reach the next vertex with even number of pendants (including 0 pendants), we change the character to label from \( b \) to \( c \) or from \( c \) to \( b \). Since the number of the vertices in the cycle with even number of pendants is even, we get a sequence like \( b, c, b, c, \cdots \) which ends at \( c \). Thus, we get a \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-magic labeling.

If the number of the vertices in the cycle with even number of pendants is odd, then, by the above analysis, the sequence \( b, c, b, c, \cdots \) must end at \( b \). But, we start from a vertex with even number of pendants. The two adjacent edges on the cycle must be labeled different. This is a contradiction. Therefore, we cannot have a \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-magic labeling. \( \square \)

Note that if the graph \( \mu(n; a_1, \cdots, a_n) \) has at least one vertex with even number of pendants, then we have both even order vertices and odd order vertices (since pendants are order 1). Therefore, it is not \( \mathbb{Z}_2 \)-magic.

**Definition 10.** Let \( \text{Gph}^* \) be the class of pairs \((H, s)\) where \( H \) is a connected graph and \( s \) is a distinguished vertex of \( H \). For any graph \( G \) and a mapping \( \Phi : V(G) \to \text{Gph}^* \), we construct a new graph, \( G \times_L \Phi \) as follows: We form the disjoint union of \( G \) and \( \{ \Phi(v) = (H, s) \mid v \in V(G) \} \) and identify \( v \) with \( s \) for each \( v \) in \( V(G) \). The resulting graph is called the generalized \( \text{L-product} \) of \( G \) with \( \Phi \).

**Definition 11.** If \( \Phi \) is the constant map of \( \Phi(v) = (H, s) \) for all \( v \in V(G) \), then we denote the \( G \times_L \Phi \) simply by writing \( G \times_L (H, s) \) and call the resulting graph the \( \text{L-product} \) of \( G \) with \( (H, s) \).

**Corollary 5.4.** The \( \text{L-products} \) of a cycle \( C_n \) with \( \text{St}(m) \) is \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-magic if and only if either \( m \) is odd or \( n \) and \( m \) are both even.

**Proof.** The \( \text{L-products} \) of a cycle \( C_n \) with \( \text{St}(m) \) is actually the womb graph \( \mu(n; m, \cdots, m) \). The result follows by the Theorems 5.2 and 5.2. \( \square \)

**References**


