On Edge-Balance Index Sets of Wheels

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Abstract

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $\mathbb{Z}_2 = \{0, 1\}$. Any edge-friendly labeling f induces a partial vertex labeling $f^+ : V(G) \to \mathbb{Z}_2$ assigning 0 or 1 to $f^+(v)$, v being an element of $V(G)$, depending on whether there are more 0-edges or 1-edges incident with v, and no label is given to $f^+(v)$ otherwise. For each $i \in \mathbb{Z}_2$, let $v_f(i) = \text{Card}\{v \in V(G) : f^+(v) = i\}$ and let $e_f(i) = \text{Card}\{e \in E(G) : f(e) = i\}$. An edge-labeling f of G is said to be edge-friendly if $\{|e_f(0) - e_f(1)| \leq 1$. The edge-balance index set of the graph G is defined as $EBI(G) = \{|v_f(0) - v_f(1)| : f \text{ is edge-friendly.}\}$. In this paper, we investigate and present results concerning the edge-balance index sets of wheels.

Keywords and phrases: vertex labeling, edge labeling, friendly labeling, cordiality, edge-balance index set, wheel.

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1 Introduction

In [3], Kong and second author considered a new labeling problem of graph theory. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and let $\mathbb{Z}_2 = \{0, 1\}$. A labeling $f : E(G) \to \mathbb{Z}_2$ induces a vertex partial labeling $f^+ : V(G) \to \mathbb{Z}_2$ defined by $f^+(v) = 0$ if the edges labeled by 0 incident on v is more than the number of edges labeled by 1 incident on v and $f^+(v) = 1$ if the edges labeled by 1 incident on v is more than the number of edges labeled by 0 incident on v. $f^+(v)$ is not defined if the number of edges labeled by 0 is equal to the number of edges labeled by 1. For $i \in \mathbb{Z}_2$, let $v_f(i) = \text{Card}\{v \in V(G) : f^+(v) = i\}$ and let $e_f(i) = \text{Card}\{e \in E(G) : f(e) = i\}$.

With these notations, we now introduce the notion of an edge-balanced graph.

Definition 1. A labeling f of a graph G is said to be ***edge-friendly*** if $|e_f(0) - e_f(1)| \leq 1$. A graph G is said to be an ***edge-balanced*** graph if there is an edge-friendly labeling f of G satisfying $|v_f(0) - v_f(1)| \leq 1$.

Chen, Lee, et al in [1] proved that all connected simple graphs except the star $K_{1,2k+1}$, where $k \geq 0$ are edge-balanced.

Definition 2. The ***edge-balance index set*** of the graph G, EBI(G), is defined as $\{|v_f(0) - v_f(1)| : \text{the edge labeling f is edge-friendly.}\}$.

We will use $v(0)$, $v(1)$, $e(0)$, $e(1)$ instead of $v_f(0)$, $v_f(1)$, $e_f(0)$, $e_f(1)$, provided there is no ambiguity.

Example 1. EBI (nK_2) is $\{0\}$ if n is even and $\{2\}$ if n is odd.

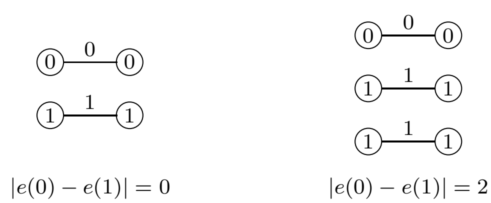


Figure 1: The edge balance index set of $2K_2$ and $3K_2$

For any $n \geq 1$, we denote the tree with $n+1$ vertices of diameter two by St(n). The star has a center c and n appended edges from c.

Example 2. The edge-balance index set of the star St(n) is

$$\text{EBI}\,(\text{St}(n)) = \begin{cases} \{0\} & \text{if } n \text{ is even,} \\ \{2\} & \text{if } n \text{ is odd.} \end{cases}$$

Example 3. In [10], Lee, Lo and Tao showed that

$$\text{EBI}\,(P_n) = \begin{cases} \{0\} & \text{if } n \text{ is } 3, \\ \{0,1\} & \text{if } n \text{ is odd and greater than } 3, \\ \{2\} & \text{if } n \text{ is } 2, \\ \{1,2\} & \text{if } n \text{ is even and greater than } 2. \end{cases}$$

Figure 2 shows the EBI of P_3 and P_4.

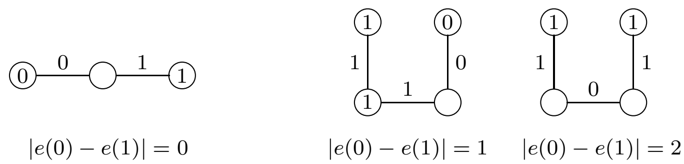


Figure 2: The edge balance index set of P_3 and P_4

Example 4. Figure 3 shows that the edge-balance index set of a tree with six vertices is $\{0, 1, 2\}$.

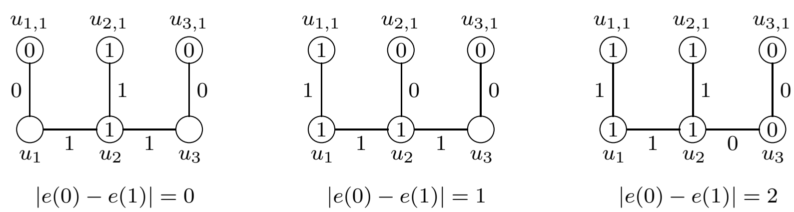


Figure 3: The edge balance index set of a tree with 6 vertices

The edge-balance index sets can be viewed as the dual of balance index sets. The balance index sets of graphs were considered in [2, 4, 6, 7, 8, 9, 11, 13]. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and let $\mathbb{Z}_2 = \{0, 1\}$. A labeling $f : V(G) \to \mathbb{Z}_2$ induces an edge partial labeling $f^* : E(G) \to A$ defined by $f^*(vw) = f(v)$, if and only if $f(v) = f(w)$ for each edge $vw \in E(G)$. For $i \in \mathbb{Z}_2$, let $v_f(i) = \text{card}\{v \in V(G) : f(v) = i\}$ and $e_{f^*}(i) = \text{card}\{e \in E(G) : f^*(e) = i\}$. A labeling f of a graph G is said to be ***friendly*** if $|v_f(0) - v_f(1)| \le 1$. If $|e_f(0) - e_f(1)| \le 1$ then G is said to be ***balanced***. The ***balance index set*** of the graph G, $\text{BI}(G)$, is defined as $\{|e_f(0) - e_f(1)| : \text{the vertex labeling } f \text{ is friendly}\}$.

Edge balance index sets of trees, flower graphs and $(p, p+1)$-graphs were considered in [5, 10, 12]. In this paper, exact values of the edge-balance index sets of wheels are presented.

2 On Edge-balance Index Sets of a Cycle

For later use, we provide here some results on the edge-balance index sets of cycles.

Notation 1. Let C_n be a cycle with a vertex set $\{c_1, c_2, \cdots, c_n\}$. Let f be an edge-labeling on C_n (not necessarily edge-friendly). We denote the numbers of edges labeled 0 or 1 by f by $e_C(0)$ or $e_C(1)$, respectively. We also denote the number of vertices labeled 0, 1, or not labeled by f^+ by $v_C(0)$, $v_C(1)$, or $v_C(\times)$, respectively.

For a vertex of order 2, with an edge-labeling (not necessarily edge-friendly), it can only be labeled in one of the following three ways.

1. If both edges are labeled 0, then the vertex is labeled 0.

2. If both edges are labeled 1, then the vertex is labeled 1.

3. If one edge is labeled 0 and another is labeled 1, then the vertex is not labeled.

If we add an edge to a vertex, then there are two cases:

A If the vertex was already labeled, then the label of the vertex is not changed after adding an edge because at least two edges are labeled by the same number.

B If the vertex was not labeled then the label of the vertex is the same as the label assigned to the new edge.

For later reference, we call these as Rule A and Rule B.

Lemma 2.1 *In a cycle C_n with an edge labeling f (not necessarily edge-friendly), we have two equations:*

$$2v_C(0) + v_C(\times) = 2e_C(0)$$

and

$$2v_C(1) + v_C(\times) = 2e_C(1).$$

Proof. Every vertex labeled 1 has two incident 1-edges and every unlabeled vertex has one incident 1-edge. No other vertex contains any edge labeled 1. Because every edge is counted twice, we have

$$2v_C(1) + v_C(\times) = 2e_C(1).$$

Similarly, we have

$$2v_C(0) + v_C(\times) = 2e_C(0).$$

□

From these two equations, we can see that $v_C(\times)$ must be even.

Corollary 2.2 *In a cycle C_n with an edge labeling f (not necessarily edge-friendly), $v_C(\times)$ is even.*

Theorem 2.3 *The edge-balance index set of a cycle C_n is*

$$EBI(C_n) = \begin{cases} \{0\} & \text{if } n \text{ is even,} \\ \{1\} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. For a friendly edge labeling f, we have $|v_C(0) - v_C(1)| \leq 1$. By Lemma 2.1, the edge-balance index is $|e_C(0) - e_C(1)| = |v_C(0) - v_C(1)|$. This completes the proof.

Example 5. Figure 4 shows that the edge-balance index sets EBI(C_3) = EBI(C_5) = {1} and EBI(C_3) = EBI(C_5) = {1}

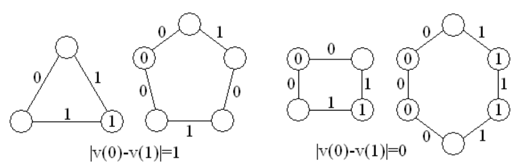


Figure 4: EBI of C_3, C_4, C_5 and C_6

Lemma 2.4 *In a cycle C_n with an edge labeling f (not necessarily edge-friendly), $e_C(0)$ or $e_C(1)$ is zero if and only if $v_C(\times) = 0$.*

Proof. If $e_C(0)$ equals zero, then all edges are labeled by 1. Thus, all vertices are labeled by 1, that is, $v_C(\times) = 0$. Similarly, if $e_C(1)$ equals zero, then $v_C(\times) = 0$.

Conversely, if both $e_C(0) > 0$ and $e_C(1) > 0$, then there is at least one 0-edge and one 1-edge in C_n. Obviously, in C_n, there must be one 0-edge which meets an 1-edge in a vertex. Thus, $v_C(\times)$ is not zero. □

Lemma 2.5 *In a cycle C_n with an edge labeling f (not necessarily edge-friendly), if $v_C(0)$ and $v_C(1)$ are both positive then $v_C(\times) > 0$.*

Proof. If both $v_C(0)$ and $v_C(1)$ are positive, then we must have at least two edges labeled 0 and two edges labeled 1. Thus, both $e_C(0)$ and $e_C(1)$ are greater than one. By Lemma 2.4, we have $v_C(\times) > 0$. □

Lemma 2.6 *In a cycle C_n with an edge labeling f (not necessarily edge-friendly), we have*

$$0 \leq v_C(0) \leq e_C(0) - 1$$

and

$$0 \leq v_C(1) \leq e_C(1) - 1.$$

Proof. Because we need two 0-edges to get a vertex labeled by 0, $e_C(0)$ 0-edges can produce at most $e_C(0) - 1$ 0-vertices. Thus,

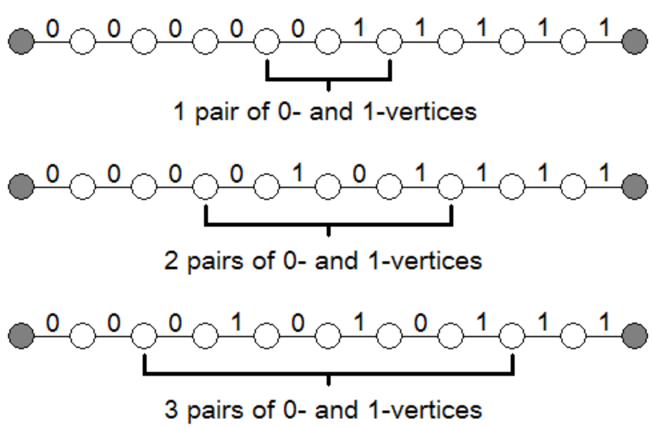
$$0 < v_C(0) < e_C(0) - 1.$$

Similarly,

$$0 < v_C(1) < e_C(1) - 1.$$

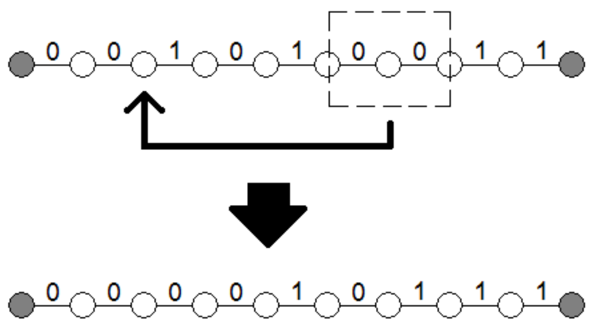
□

By Corollary 2.2, we know that $v_C(\times)$ must be even. For an even number $2k > 0$, we can construct an edge labeling of C_n with exactly $2k$ unlabeled vertices. First, put k pairs of 0- and 1-edges alternately in the middle of P_{n+1} and all the remaining 0s on the left side edges and all the remaining 1s on the right side edges. Then, glue the two sided vertices together. (See the following figure.)



Since we put k pairs of 0 and 1 alternately, these $2k$ edges produce exactly $2k-1$ unlabeled vertices. Additionally, the glued two sided vertices become an unlabeled vertex. All the other vertices are either in the middle of 0-edges or 1-edges. Thus, we have exactly $2k$ unlabeled vertices.

For an edge labeling f of C_n with $v_C(\times) = 2k > 0$, we can rearrange it into an edge labeling we just constructed without altering $v_C(\times)$. Pick an unlabeled vertex and split this vertex into two vertices. It becomes a P_{n+1} with the very left edge labeled by 0 and the very right edge labeled by 1. Let a maximal chain of 0-edges be a path with only 0-edges where its length is greater than 1 and it is not a subchain of 0-edges with a longer length. If a maximal chain of 0-edges does not contain the very left 0-edge, we can cut it off and insert it into the left side of the first from the left unlabeled vertex without altering the number of 0-vertices, 1-vertices and unlabeled vertices. (See the following figure.)



By repeating this process, it will become the one we constructed previously.

Lemma 2.7 *In a cycle C_n with an edge labeling f (not necessarily edge-friendly), assume that $v_C(\times) = 2k > 0$. Then*

$$v_C(1) = e_C(1) - k.$$

Proof. Since $v_C(\times) = 2k > 0$, by the above rearrangement, we can collect k pairs of 0- and 1-edges in the middle of P_{n+1} without altering $v_C(\times) = 2k$. Since these k pairs of edges occupy k edges labeled 1, there are only $e_C(1)-k$ 1-edges left on the right part of P_{n+1}. Since the very right edge of the k pairs of 0- and 1-edges is an edge labeled 1, we have a chain of 1-edges with $e_C(1) - k + 1$ edges and $e_C(1) - k$ vertices in between edges. Thus,

$$v_C(1) = e_C(1) - k.$$

□

Corollary 2.8 *In a cycle C_n with an edge labeling f (not necessarily edge-friendly), assume that $v_C(\times) = 2k > 0$. Then*

$$v_C(0) = n - e_C(1) - k.$$

Proof. By Lemma 2.7, we have

$$v_C(1) = e_C(1) - k.$$

In C_n, we have $n = v_C(0) + v_C(1) + v_C(\times)$. Thus,

$$\begin{aligned} v_C(0) &= n - v_C(1) - v_C(\times) \\ &= n - (e_C(1) - k) - 2k \\ &= n - e_C(1) - k. \end{aligned}$$

□

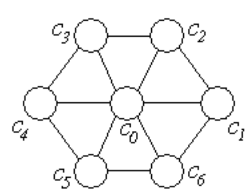
Note that since every unlabeled vertex requires one 0-edge and one 1-edge, $v_C(\times) = 2k > 0$ unlabeled vertices require k 1-edges. This leads us to

Lemma 2.9 *In a cycle C_n with a labeling f (not necessarily edge-friendly) where the number of unlabeled vertices $v_C(\times) = 2k > 0$. We have*

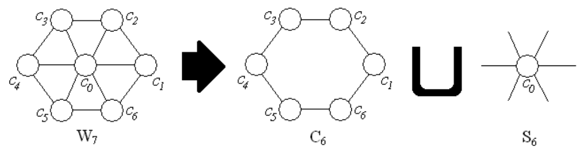
$$1 \leq k \leq e_C(1).$$

3 On Edge-balance Index Sets of Wheels

For $n \geq 4$, the ***wheel*** on n vertices, W_n, is a graph with n vertices $\{c_0, c_1, c_2, \cdots, c_{n-1}\}$, where c_0 is of degree $n-1$ and all the other vertices are of degree 3. It is the cycle C_{n-1} with an additional vertex c_0 connected to each vertex of C_{n-1}. The vertex c_0 is called the hub, and the edges connecting the hub to the other vertices are called the spokes. W_7 is displayed below:



By the nature of a wheel W_n, we split a wheel into two pieces, a cycle C_{n-1} outside and a star inside. The star part, denoted by S_{n-1}, contains the center vertex c_0 and $n-1$ edges. Note that S_{n-1} is not a subgraph of W_n. One can see W_7 here as an example.



We now use the notations from section 2. Let f be an edge-friendly labeling of W_n. We denote the number of edges of C_{n-1} which are labeled 0 and 1 by f^+ by $e_C(0)$ and $e_C(1)$ and the number of edges in S_{n-1} which are labeled 0 and 1 by f^+ by $e_S(0)$ and $e_S(1)$.

S_{n-1} has only one vertex. Let δ be the value of balance index in the form of $v(0)-v(1)$ of this vertex. Thus,

$$\delta = \begin{cases} 1 & \text{if } c_0 \text{ is labeled by } 0, \\ 0 & \text{if } c_0 \text{ is not labeled}, \\ -1 & \text{if } c_0 \text{ is labeled by } 1. \end{cases}$$

If we focus on C_{n-1}, then the restriction of f on C_{n-1} is an edge labeling of C_{n-1}. Thus, all the results in section 2 are applicable here. Therefore, we still denote the number of vertices C_{n-1} labeled 0, 1, and not labeled by the restricted f^+ by $v_C(0)$, $v_C(1)$, and $v_C(x)$, respectively.

When we put S_{n-1} into C_{n-1} to get our wheel back, the labels of vertices of C_{n-1} change. To distinguish before and after, we name the number of vertices C_{n-1} in W_n labeled 0 or 1 by the original f^+ by $v_W(0)$ or $v_W(1)$, respectively.

By the above notations, we have

Lemma 3.1 *Let W_n be a wheel. The balance index is*

$$v(0) - v(1) = v_W(0) - v_W(1) + \delta.$$

By the symmetry of the role of 0 and 1 in the labeling, to calculate the edge balance index, without loss of generality, we may assume that

$$e_C(0) \geq e_C(1) \geq 0.$$

Since $e_C(0) \geq e_C(1)$ and $e_C(0) + e_C(1) = n-1$, we can find the range of possible values of $e_C(1)$ as follow:

1. $0 \leq e_C(1) \leq \frac{n-1}{2}$ if n is odd, or,

2. $0 \le e_C(1) \le \frac{n}{2} - 1$ if n is even.

In C_{n-1}, under the assumption $e_C(0) \ge e_C(1) \ge 0$, there are three possible cases of the relationship between the values of $e_C(0)$ and $e_C(1)$:

1. $e_C(0) > e_C(1) \ge 1$, or,
2. $e_C(0) = e_C(1) \ge 1$ only when n is odd (which implies that $e_C(0) = e_C(1) = \frac{n-1}{2}$), or,
3. $e_C(1) = 0$ (which implies that $e_C(0) = n - 1$.)

For case 1, by Lemma 2.4, $v_C(\times)$ is not zero. Since $v_C(\times)$ must be even, we assume that $v_C(\times) = 2k > 0$ where k is a positive integer. By Lemma 2.7 and Corollary 2.8, we have

$$v_C(1) = e_C(1) - k$$

and

$$v_C(0) = (n-1) - e_C(1) - k.$$

After putting all S_{n-1} into C_{n-1} to get our wheel back, all vertices of C_{n-1} become of order 3. Thus, the Rule A and Rule B in section 2 apply. Therefore, by Rule B, all unlabeled vertices in C_{n-1} become labeled by either 0 or 1. Also, by Rule A, all other labeled vertices remain labeled by the same value. Assume that there are h 0-edges in S_{n-1} connected to an unlabeled vertex of C_{n-1}. It is clear that

Lemma 3.2 *In a wheel W_n, assume that there are h 0-edges in S_{n-1} connected to an unlabeled vertex of C_{n-1}. Then, we have*

$$0 \le h \le e_N(0).$$

If $e_S(0) \le v_C(\times)$, then we have h 0-vertices converted from unlabeled vertices and the rest $v_C(\times) - h$ unlabeled vertices are converted into vertices labeled by 1. Therefore, the number of vertices labeled by 0 is

$$v_W(0) = v_C(0) + h$$

and the number of vertices labeled by 1 is

$$v_W(1) = v_C(1) + v_C(\times) - h.$$

So, by Lemma 3.1, the edge-balance index

$$v(0) - v(1) \quad = \quad (v_C(0) + h) - (v_C(1) + v_C(\times) - h) + \delta$$

$$
\begin{aligned}
&= v_C(0) + 2h - v_C(1) - 2k + \delta \\
&= (n - 1 - e_C(1) - k) + 2h - (e_C(1) - k) - 2k + \delta \\
&= n - 1 - 2(e_C(1) + k - h) + \delta.
\end{aligned}
$$

If $v_C(\times) < e_S(0)$, then h must be less than $v_C(\times)$ since those are all unlabeled vertices available to be connected by an 0-edge. We have a similar inequality

$$0 \leq h \leq v_C(\times) < e_C(1).$$

In the calculation of edge-balance index set, this case will be overridden by the previous one since we have to calculate the balance index for all possible edge friendly labelings. Thus, it will not generate any new edge-balance index.

With Lemmas 2.9 and 3.2, we conclude that

Lemma 3.3 *In a wheel W_n, assume that $v_C(x) = 2k > 0$ where k is a positive integer and $e_C(0) > e_C(1)$. Let h be the number of 0-edges in the part of S_{n-1} connected to an unlabeled vertex of C_{n-1}. Then, the balance index is*

$$v(0) - v(1) = n - 1 - 2(e_C(1) + k - h) + \delta,$$

where $1 \leq k \leq e_C(1)$ and $0 \leq h \leq e_N(0)$ with

$$0 \leq e_C(1) \leq \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd, or,} \\ \frac{n}{2} - 1 & \text{if } n \text{ is even.} \end{cases}$$

For case 2, the argument is very similar to that of case 1. By Lemma 2.7 and Corollary 2.8, we get similar equations

$$v_W(0) = v_C(0) + h$$

and

$$v_W(1) = v_C(1) + v_C(\times) - h.$$

Since $e_C(0) = e_C(1) = \frac{n-1}{2}$, by Lemma 3.1, the balance index is

$$
\begin{aligned}
v(0) - v(1) &= (v_C(0) + h) - (v_C(1) + v_C(\times) - h) + \delta \\
&= v_C(0) + 2h - v_C(1) - 2k + \delta \\
&= (n - 1 - e_C(1) - k) + 2h - (e_C(1) - k) - 2k + \delta \\
&= n - 1 - 2(\frac{n-1}{2} + k - h) + \delta \\
&= 2(h - k) + \delta
\end{aligned}
$$

Because $e_C(1) = \frac{n-1}{2}$, by Lemma 2.9, we have $0 < k \leq \frac{n-1}{2}$.

For h, if $0 < k \leq \frac{n-1}{4}$, since $e_N(0) > 2k$ which provides enough 0-edges to be used, we have $0 \leq h \leq \frac{n-1}{2}$. Also, if $\frac{n-1}{4} \leq k \leq \frac{n-1}{2}$, since the number of unlabeled vertices is greater or equal to $e_S(1)$, h is restricted on $k - \frac{n-1}{4} \le h \le \frac{n-1}{2}$.

This discussion leads us to state the following:

Lemma 3.4 *In a wheel W_n, assume that $v_C(x) = 2k > 0$ where k is a positive integer and $e_C(0) = e_C(1)$. Let h be the number of 0-edges in the part of S_{n-1} connected to an unlabeled vertex of C_{n-1}. Then, the balance index is*

$$v(0) - v(1) = 2(h - k) + \delta,$$

where $1 \le k \le \frac{n-1}{2}$ *and* $\begin{cases} 0 \le h \le \frac{n-1}{2} & \text{if } 0 < k \le \frac{n-1}{4}, \text{ or,} \\ k - \frac{n-1}{4} \le h \le \frac{n-1}{2} & \text{if } \frac{n-1}{4} \le k \le \frac{n-1}{2}. \end{cases}$

For case 3, since all the edges in C_{n-1} are labeled by 0, we have $n-1$ vertices labeled by 0. By Rule A, the edge-balance index is

$$v_W(0) - v_W(1) = n - 1.$$

Lemma 3.5 *In a wheel W_n with $v_C(x) = 0$, the balance index is*

$$v(0) - v(1) = n - 1 + \delta,$$

Now, we are in a position to determine the balance index sets of W_n.

Theorem 3.6 *If n is even, then*

$$EBI(W_n) = \{0, 2, \cdots, 2i, \cdots, n - 2\}.$$

Proof. When n is even, we have only two cases, 1 and 3.

Since the number of edges of our wheel W_n is even, (actually, it is $2n-2$,) and f is an edge-friendly labeling of W_n, we have

$$e_S(0) = e_C(1)$$

and

$$e_S(1) = e_C(0).$$

For (a), by the assumption that $e_C(0) > e_C(1)$, we have $e_S(1) > e_S(0)$. This implies that c_0 must be labeled by 1. Thus, $\delta = -1$.

By Lemma 3.3 and $\delta = -1$, we have

$$v(0) - v(1) = n - 1 - 2(e_C(1) + k - h) + \delta = n - 2 - 2(e_C(1) + k - h)$$

where $0 < k \leq e_C(1)$ and $0 \leq h \leq e_C(1)$ with $0 < e_C(1) < \frac{n}{2}$. This implies that $1 \leq e_C(1) + k - h \leq 2e_C(1) < n$. Let $t = e_C(1) + k - h$. Since k can be any integer between 1 and $e_C(1)$ including 1 and $e_C(1)$ and h can be any integer between 0 and $e_C(1)$ including 0 and $e_C(1)$, we have $1 \leq t < n$. Thus, $v(0) - v(1) = n - 2 - 2t$ where $1 < t < n - 1$ where t is an integer. Thus, $-(n-2) < v(0) - v(1) < n - 4$ and $v(0) - v(1)$ must be even. Therefore, $\{0, 2, \cdots, 2i, \cdots, n-2\}$ is a subset of EBI(W_n).

For (c), by the assumption that $e_C(1) = 0$, we have $e_S(0) = 0$ and $e_S(1) = n - 1$. This implies that c_0 must be labeled by 1. Thus, $\delta = -1$. By Lemma 3.5,

$$v(0) - v(1) = n - 1 + \delta = n - 2.$$

Since when n is even, case 1 and case 3 are the only two cases, the balance index set of W_n is

$$\text{EBI}(W_n) = \{0, 2, \cdots, 2i, \cdots, n-2\} \bigcup \{n-2\} = \{0, 2, \cdots, 2i, \cdots, n-2\}.$$

□

Example 6. Figure 5 shows that EBI$(W_6) = \{0, 2, 4\}$.

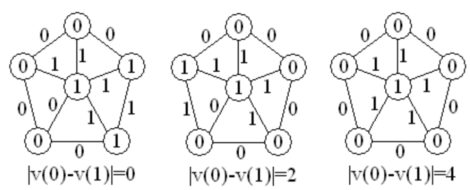


Figure 5: EBI(W_6)

Example 7. Figure 6 shows that EBI$(W_8) = \{0, 2, 4, 6\}$.

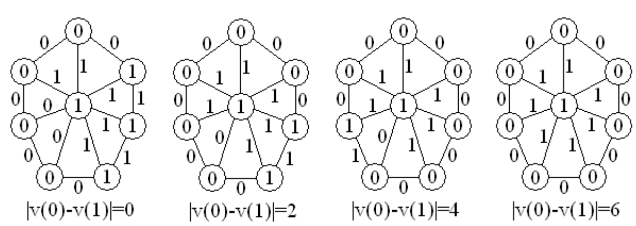


Figure 6: EBI(W_8)

Theorem 3.7 *If n is odd, then*

$$EBI(W_n) = \{1, 3, \cdots, 2i+1, \cdots, n-2\} \cup \{0, 1, 2, \cdots, \frac{n-1}{2}\}.$$

Proof. Since the number of edges of our wheel W_n is even, (actually, it is $2n-2$,) and f is an edge-friendly labeling of W_n, we have

$$e_S(0) = e_C(1)$$

and

$$e_S(1) = e_C(0).$$

For cases 1 and 3, an argument similar to the one in Theorem 3.6 applies here. Thus, we have $v(0) - v(1) = n - 2 - 2t$ where $1 \leq t \leq n-2$, being t is an integer. This implies that $-(n-2) \leq v(0) - v(1) \leq n-4$. Note that $v(0) - v(1)$ must be odd since n is odd. Therefore, EBI(W_n) contains $\{1, 3, \cdots, 2i+1, \cdots, n-2\}$.

For case 2, since $e_C(0) = e_C(1) = \frac{n-1}{2}$, c_0 is not labeled because $e_N(0) = e_C(1) = \frac{n-1}{2} = e_C(0) = e_N(1)$. By Lemma 3.4, we have $v(0) - v(1) = 2(h-k)$ where $0 < k \leq \frac{n-1}{2}$ and $0 \leq h \leq \frac{n-1}{2}$ if $0 < k \leq \frac{n-1}{4}$ and $k - \frac{n-1}{2} \leq h \leq \frac{n-1}{2}$ if $\frac{n-1}{4} \leq k \leq \frac{n-1}{2}$. Because h and k are integers, we have $-\frac{n-1}{2} \leq v(0) - v(1) = 2(h-k) \leq \frac{n-1}{2}$. Therefore, EBI($W_n$) contains $\{0, 1, 2, \cdots, \frac{n-1}{2}\}$.

With all three cases together, the balance index set of W_n is

$$\begin{aligned} & \text{EBI}(W_n) \\ = \ & \{1, 3, \cdots, 2i+1, \cdots, n-2\} \bigcup \left\{0, 1, 2, \cdots, \frac{n-1}{2}\right\} \bigcup \{n-2\} \\ = \ & \{1, 3, \cdots, 2i+1, \cdots, n-2\} \bigcup \left\{0, 1, 2, \cdots, \frac{n-1}{2}\right\}. \end{aligned}$$

Example 8. We list the edge-balance index set of W_n when n is odd and small.

1. EBI(W_5) = $\{0, 1, 2, 3\}$;
2. EBI(W_7) = $\{0, 1, 2, 3, 5\}$;
3. EBI(W_9) = $\{0, 1, 2, 3, 4, 5, 7\}$;
4. EBI(W_{11}) = $\{0, 1, 2, 3, 4, 5, 7, 9\}$;
5. EBI(W_{13}) = $\{0, 1, 2, 3, 4, 5, 6, 7, 9, 11\}$;

6. $\mathrm{EBI}(W_{15}) = \{0, 1, 2, 3, 4, 5, 6, 7, 9, 11, 13\}$;

7. $\mathrm{EBI}(W_{17}) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 15\}$;

Example 9. Figure 7 shows that $\mathrm{EBI}(W_7) = \{0, 1, 2, 3, 5\}$. Note that 4 is missing.

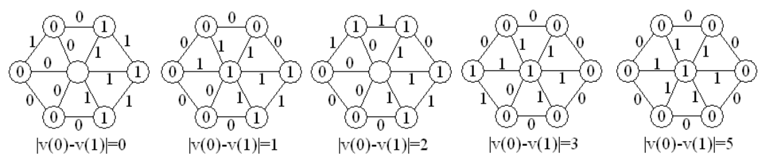


Figure 7: $\mathrm{EBI}(W_7)$

Example 10. Figure 8 shows that $\mathrm{EBI}(W_{11}) = \{0, 1, 2, 3, 5, 7, 9\}$. Note that 6 and 8 are missing.

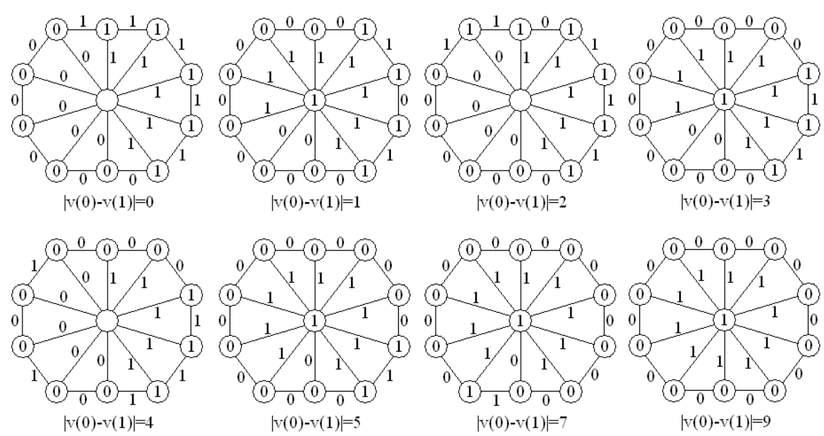


Figure 8: $\mathrm{EBI}(W_{11})$

References

[1] B.L. Chen, K.C. Huang, S.M. Lee and S.S. Liu, On edge-balanced multigraphs, *Journal of Combinatorial Mathematics and Combinatorial Computing*, **42** (2002), 177-185.

[2] S.R. Kim, S.M. Lee and H.K. Ng, On balancedness of some graph constructions, *Journal of Combinatorial Mathematics and Combinatorial Computing*, **66** (2008), 3-16.

[3] M.C. Kong and S.M. Lee, On Edge-Bananced Graphs, in *Proceedings of the* 7*th quadrennial International conference on the theory and applications of graphs*, vol **2**, 712-722, John Wiley and Sons, Inc. 1993

[4] H. Kwong and S.M. Lee, On balance index sets of chain sum and amalgamation of generalized theta graphs, *Congressus Numerantium*, **187** (2007), 21-32.

[5] H. Kwong and S.M. Lee, On edge-balance index sets of flower graphs, unpublished manuscript.

[6] H. Kwong, S.M. Lee and D.G. Sarvate, On balance index sets of one-point unions of graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing*, **66** (2008), 113-127.

[7] A.N.T. Lee, S.M. Lee and H.K. Ng, On balance index sets of graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing*, **66** (2008), 135-150.

[8] S.M. Lee, B. Chen, and T. Wang, On the balanced windmill graphs, *Congressus Numerantium*, **186** (2007), 9-32.

[9] S.M. Lee, A.C. Liu and S.K. Tan, On balanced graphs, *Congressus Numerantium*, **87** (1992), 59-64.

[10] S.M. Lee, S.P.B. Lo and M.F. Tao, On the edge-balance index sets of some trees, unpublished manuscript.

[11] S.M. Lee, H.K. Ng and S.M. Tong, On the balance index set of the chain-sum graphs of cycles, *Utilitas Mathematica*, **77** (2008), 113-123.

[12] S.M. Lee, H.H. Su and Y.C. Wang, On the edge-balance index sets of $(p, p+1)$-graphs, unpublished manuscript

[13] M.A. Seoud and A.E.I. Abdel Maqsoud, On cordial and balanced labelings of graphs, *Journal of Egyptian Mathematics Society*, **7** (1999) 127-135.