On Computing Edge-Magic Graphs and Q(a) Balance Edge-Magic Graphs

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Abstract If G is a (p,q)-graph in which the edges are labeled 1,2,3,...q so that the vertex sums are constant, mod p, then G is called an ***Edge-Magic graph*** (in short, ***EM graph***). A (p,q)-graph G in which the edges are labeled by Q(a) so that the vertex sums mod p is a constant, is a called ***Q(a) Balance Edge-Magic graph*** (in short, ***Q(a)-BEM graph***), where for a ≥ 1, we denote

$$Q(a)=\begin{cases}\{\pm a\ \ldots,\ \pm(a-1+q/2)\}, & \text{if q is even,}\\ \{0,\ \pm a,\ \ldots,\ \pm(a-1+(q-1)/2)\}, & \text{if q is odd.}\end{cases}$$

Our purpose of this paper is to show that the theory of EM graphs and the theory of Q(a)-BEM graphs are unrelated. Particularly, we investigate that some graphs are both EM and Q(a)-BEM graphs; Some Q(a)-BEM graphs which are not EM; Infinitely many Non-Q(a)-BEM graphs which are not EM. Several conjectures are proposed. Finally, we show that both the EM graph problem and Q(a)-BEM graph problem are NP-hard.

Keywords: Edge-magic, Q(a) balance edge-magic, Strong BEM, NP hard.

1. Introduction.

All graphs in this paper are connected (multi-) graphs without loops.

Definition 1.1. Let G be a (p,q)-graph in which the edges are labeled 1, 2,...,q. The vertex sum for a vertex v is the sum of the labels of the incident edges at v. If the vertex sums are constant, mod p, then G is called an **Edge-Magic graph** (for simplicity we denote **EM graph**).

The concept of EM graphs was introduced by the Lee, Seah and Tan [5].

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Lemma 1.1. A necessary condition for a (p,q)-graph to be Edge Magic is $q(q+1) \equiv 0 \pmod{p}$.

Example 1. Table 1 shows that a 3×3 matrix, all its row sum, column, and diagonal sum are 15. We could construct a graph $G_1 = K(3,3)$ with 6 vertices and 9 edges that is EM with constant sum 15, mod 6 = 3. (See Figure 1.) Note that this (6,9)-graph satisfies the necessary condition $9(9+1) \equiv 0 \pmod{6}$.

2	7	6
9	5	1
4	3	8

Table 1.

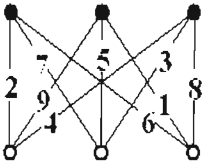
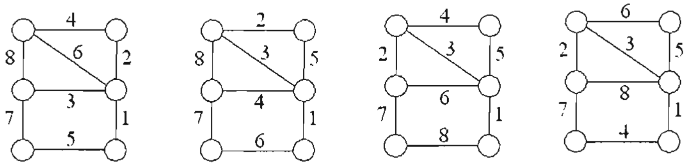


Figure 1. G_1.

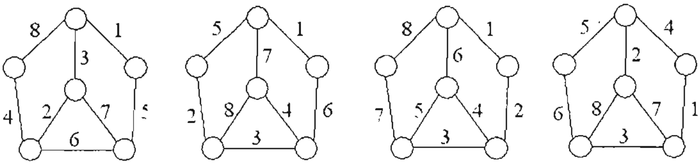
Example 2. Figure 2 shows a graph G_2 with 6 vertices and 8 edges that is EM with different constant sums.



c = 0 (mod 6) c = 1 (mod 6) c = 3 (mod 6) c = 5 (mod 6)

Figure 2. G_2.

Example 3. Figure 3 shows another graph G_3 with 6 vertices and 8 edges that is EM with different constant sums.



c = 0 (mod 6) c = 1 (mod 6) c = 3 (mod 6) c = 5 (mod 6)

Figure 3. G_3.

Example 4. Figure 4 shows another graph G_4 with 6 vertices and 9 edges that is EM with different constant sums.

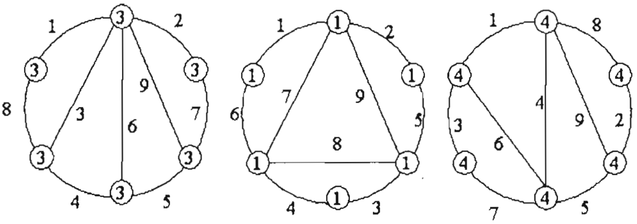


Figure 4. G_4.

Remark 1.1. However, there are infinitely many connected graphs such as trees and cycles that satisfy the necessary condition but they are not EM.

Definition 1.2. A (p,q)-graph G in which the edges are labeled by Q(a) so that the vertex sums mod p is a constant, is called **Q(a)-Balance Edge-Magic** (in short, **Q(a)-BEM**), where For $a \geq 1$, we denote

$$Q(a)=\begin{cases} \{\pm a \ldots, \pm(a-1+q/2)\}, \text{ if q is even,} \\ \{0, \pm a, \ldots, \pm(a-1+(q-1)/2)\}, \text{ if q is odd.} \end{cases} \quad (1.1)$$

Example 5. (**Q(1)-BEM**) The following (3,9)-multigraph graph G_5 is Q(1)-BEM.

Remark 1.2. G_5 is EM since we could partition the edge labels into three group sets: {2, 4, 9}, {1,6, 8}, and {3, 5,7}, so that the vertex sums are 30, mod 3 =0.

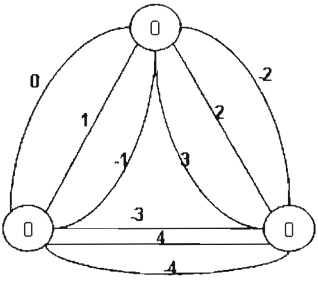


Figure 5. G_5

Example 6. (**Q(1)-BEM**) The following (4,5)-graph G_6 is Q(1)-BEM.
Two different Q(1)-BEM labelings with sum 1 and 3 shown in Figure 6(A) and 6(B), respectively.

Remark 1.3. G_6 is not EM.

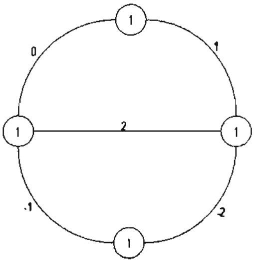


Figure 6(A). G_6.

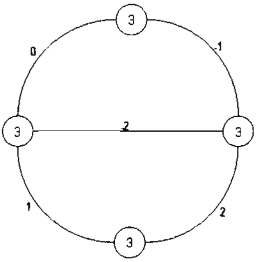


Figure 6(B). G_6.

Example 7. **(Q(1)-BEM, Q(2)-BEM)** The following graph G_7 is Q(1)-BEM, Q(2)-BEM.

Remark 1.4. G_7 is also EM.

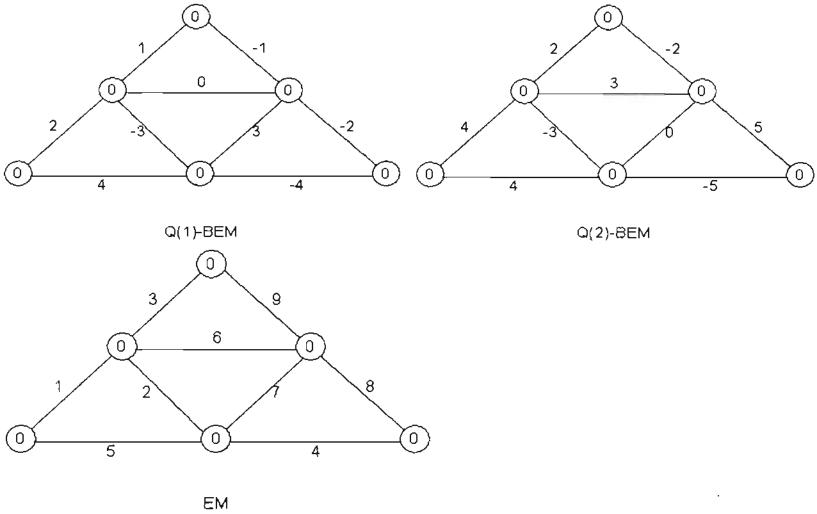
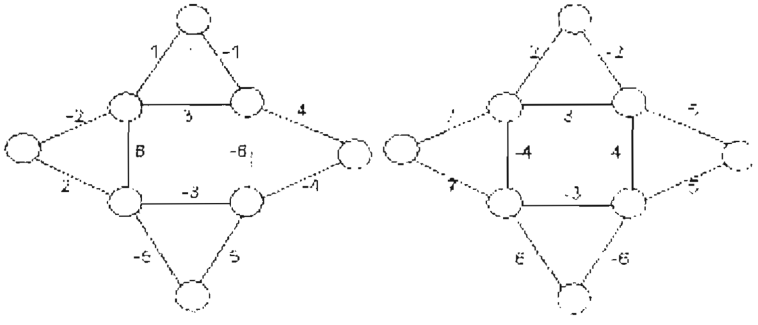


Figure 7. G_7.

Example 8. **(Q(1)-BEM, Q(2)-BEM)** The following graph G_8 is Q(1)-BEM and Q(2)-BEM.

Remark 1.5. G_8 is not EM.



Q(1)-BEM Q(2)-BEM

Figure 8. G_8.

Example 9. The following two (6,7)-graphs are Q(a)-BEM graphs for a =6x+5, where x =0,1,2,3,..... Figure 9(A), 9(B) shows the Q(a)-BEM labelings for these two graphs, G_{9A}, and G_{9B}, respectively.

Remark 1.6. G_{9A}, G_{9B} are not EM graphs.

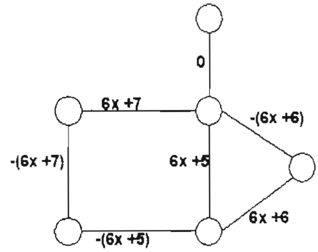


Figure 9(A). G_{9A}

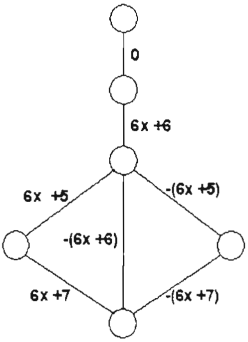


Figure 9(B). G_{9B}

Definition 1.3. A graph G is **strong BEM** if it is Q(a)-BEM for all $a \geq 1$.

By the Definition of Q(a)-BEM, we have the following result:

Theorem 1.2. If a regular (p,q)-graph is Q(a)-BEM then it is Q(kp+a)-BEM for all $k \geq 1$.

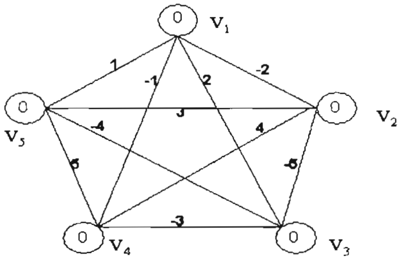
Corollary 1.3. If a (p,q)-graph is Q(a)-BEM for a =1,2,...,p-1 then it is strong BEM.

Theorem 1.3. The complete graph $G_{10} = K_5$ is EM and it is strong BEM.
Proof: We first show that G_{10} is EM.
Suppose we let the vertices in $G_{10} = K_5$, be v_1, v_2, v_3, v_4, v_5 in clockwise order. Then the following labeling matrix in Table 2, it shows that $G_{10} = K_5$ is EM. !

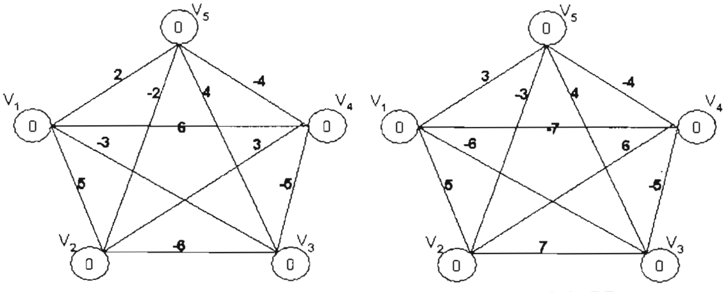
	v_1	v_2	v_3	v_4	v_5
v_1		1	7	9	5
v_2	1		2	6	8
v_3	7	2		3	10
v_4	9	6	3		4
v_5	5	8	10	4	

Table 2. $G_{10} = K_5$ is EM.

It suffices to show that $G_{10} = K_5$ is Q(a)-BEM for a=1,2,3,4,5 in Figure 10. !



K 5 is Q(1)-BEM



K 5 is Q(2)-BEM K 5 is Q(3)-BEM

Figure 10. $G_{10} = K_5$ is strong BEM.

2. The theory of EM graphs and the theory of Q(a)- BEM graphs are unrelated

Our purpose of this paper is to show that the theory of EM graphs and the theory of Q(a)-BEM graphs are unrelated. Particularly, we investigate that some graphs are both EM and Q(a)-BEM graphs; Some Q(a)-BEM graphs which are not EM; Infinitely many Non-Q(a)-BEM graphs which are not EM.

(2A) Some graphs are both EM and Q(a)-BEM

Note that Graphs G_1, G_5, G_7 G_{13} shown in Examples 1, 5, 7 are both EM and Q(a)-BEM.

Theorem 2.1. The complete bipartite graph $G_1 = K(3,3)$ is EM and Q(1)-BEM.
Proof. (See also Example 1, and Table 3(A), $G_1 = K(3,3)$ is **EM**).
Note that $G_1 = K(3,3)$ has 6 vertices and 9 edges. Suppose the vertices on the top, from left to right, be u_1, u_2, u_3 and the vertices on the bottom, from left to right, be v_1, v_2, v_3. We provide a labeling matrix in Table 3(A) for G_1, we see that it is EM. Another labeling matrix in Table 3(B) for G_1, it shows that G_1 is Q(1)-BEM. !

	v_1	v_2	v_3
u_1	2	3	7
u_2	4	5	9
u_3	6	10	8

Table 3(A).

	v_1	v_2	v_3
u_1	-4	-3	1
u_2	-2	-1	3
u_3	0	4	2

Table 3(B).

(2B) Some Q(a)-BEM graphs which are not EM

Note that Graphs G_6, G_8, G_{9A}, G_{9B} G_{10}, G_{11}, G_{12} shown in Examples 6, 8, 9, 10, 11, 12 are Q(a)-BEM, but not EM.

Example 11. Figure 11 shows a graph G_{11} with 5 vertices and 6 edges that is not EM; however, it is Q(1)-BEM; but, it is neither Q(2) BEM, nor Q(3)-BEM.

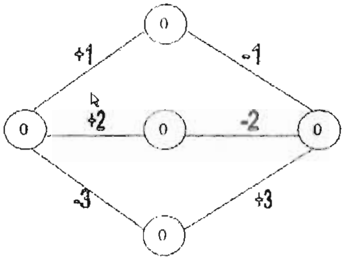


Figure 11. G_{11}.

For n >3, the *wheel* on *n* vertices, W_n is a graph with *n* vertices $x_1, x_2, \ldots, x_n$, x_1 having degree *n-1* and all the other vertices having degree *3*. The vertex x_1 is adjacent to all the other vertices, and for $i=2,\ldots,n-1$, x_i is adjacent to x_{i+1}, and x_{n-1} is adjacent to x_2. The edges of a wheel which include the hub are called spokes . In a wheel graph, the hub has degree n-1, and other vertices have degree 3. See Figure 12 for W_4, W_5, W_6, W_7 .

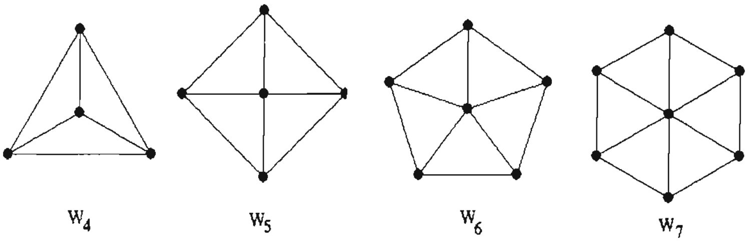


Figure 12. G_{12}.

Theorem 2.2. All wheels are not EM.
Proof. Note that $p(W_n) = n$ and $q(W_n) = 2n-2$. It is clear that $q(q+1) \bmod n \neq 0$. Thus, W_n is not EM. !

S. M. Lee, et al [10] showed the following Q(a)-BEM results for wheels.

Theorem 2.3. The wheel W_4 is Q(a)-BEM if and only if $a = 3 \pmod 4$.

Theorem 2.4. The wheel W_6 is Q(a)-BEM if and only if $a = 1,2$ (mod 6).
Theorem 2.5 The wheel W_5, W_7, W_8 are strong BEM.
Theorem 2.6 The wheel W_9 is Q(a)-BEM if and only if a=1,2,3,4,5,6.

(2C) Infinitely many Non-Q(a)-BEM graphs which are not EM

Theorem 2.7 [10]. A necessary condition for a (p,q)- graph with two degree one vertices u_1 and u_2, to be **Non-Q(a)-BEM** for any a > 1 is
$\lceil (q/2) \rceil < p$.
Corollary 2.8. All trees except P_2 are not Q(a)-BEM for all a >1.

Remark 2.1. By Theorem 1.1 and 2.7, we see that there are Infinitely many Non-Q(a)-BEM graphs, which are also not EM.

Remark 2.2. There following (8,17)-graph which does not has $\lfloor q/2 \rfloor <p$, however it is Q(1)-BEM and it is not EM.

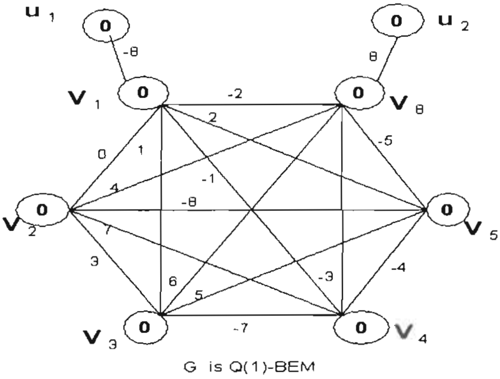


Figure 13. G_{13}.

We have investigated that some graphs are both EM and Q(a)-BEM graphs; some Q(a)-BEM graphs which are not EM; some non-Q(a)-BEM graphs. Therefore, in conclusion, we claim that the theory of EM graphs and the theory of Q(a)-BEM graphs are unrelated.

Note that there are more complete results on EM, Q(a)-BEM graphs are provided in [10]. We listed results for the complete graphs and complete bipartite graphs for the completeness of our discussions.

Theorem 2.9. The complete graph K_n is Q(a)-BEM for a = 1, 2, ….,n-1, for n=5, 6, 7, 8. Thus, K_n is strong BEM, for n=5, 6, 7, 8.

Theorem 2.10. A complete bipartite graph K(2,m) is EM if and only if m=4 or 10; Also, it is Q(1)-BEM for all odd m integers.
Remark 2.3. K(2,m) is not Q(1)-BEM for m = 2,6. But it is Q(1)-BEM for m = 4,8.

Theorem 2.11. The complete bipartite graphs K(2,4), K(3,4) are strong-BEM.

Theorem 2.12. The complete bipartite graphs K(2,8), K(3,3) are Q(1)-BEM.

3. Some Conjectures

We propose the following conjectures for further research.

Conjecture 3.1. The complete graph K_n is strong BEM for n $\geq$ 9.

Conjecture 3.2. The wheel W_n is strong BEM for n $\geq$ 10.

4. The EM Graph Problem and Q(a)-BEM Graph Problem are NP-hard.

Finding the EM and Q(a)-BEM labelings of graphs are related to solving system of linear Diophantine equations. In general, it is difficult to find an EM labeling or a Q(a)-BEM labeling of a graph. For the following, we show that both the EM graph problem and Q(a)-BEM graph problem are NP-hard.

Theorem 4.1.
Edge-Magic Graph Labeling Problem is NP-hard.

Proof. First, we formulte ***the Edge-Magic Graph Labeling Problem*** as follows.
INSTANCE: Given Graph G = (V,E), $|V| = p, |E| = q$, weights $w(v) \in Z^+$ be vertex sum for each $v \in V$ and $l(e) \in \{1, 2, \ldots q\} \in Z^+$ for each $e \in E$, positive integer $K = \sum_{i=1}^{q} i = \frac{q(1+q)}{2}$ (i.e., the maximum vertex sums) and $J = \sum_{i=1}^{p} i = \sum (1 + 2 + \ldots p)$.

QUESTION: Is there a partition of V into disjoint $V_1, V_2, \ldots, V_p$ such that $w(v) = k + pm \leq K$, for $v \in V_i$, $1 \leq i \leq p$, where m, $0 \leq k \leq p-1$ are integers such that $\sum_{e \in E} l(e) \leq J$?

In [16, p209], we know that ***the Graph Partitioning Problem*** is NP-Complete (See Appendix.). By the above formulation, we could perform ***Polynomial Transformation*** from the Graph Partitioning problem into the Edge-Magic Graph Labeling Problem. Thus, the Edge-Magic Graph Labeling Problem is NP-hard. "

Theorem 4.2.
Q(a)-Balanced Edge-Magic Graph Labeling Problem is NP-hard.

Proof. Recall that a (p,q)-graph G in which the edges are labeled by Q(a) so that the vertex sums mod p is a constant, is a called ***Q(a)-BEM graph***, where for a $\geq$ 1, we denote Q(a) in (1.1), that is,

$$Q(a) = \begin{cases} \{\pm a \ldots, \pm(a-1+q/2)\}, & \text{if q is even,} \\ \{0, \pm a, \ldots, \pm(a-1+(q-1)/2)\}, & \text{if q is odd.} \end{cases}$$

First, we observe that the Q(a) labels include both positive and negative integers. Note that we say that the intergers ***a and b are congruent modulo m*** if only if there is an interger k such that $a = b+km$, where m is a positive integer. Therefore, for a negative integer $-a$, we could find a positive integer a' such that *a' and $-a$ are congruent modulo p* if only if there is an integer k such that $a' = (-a)+kp$, where k is an integer. In this way, we could construct a set Q'(a) from Q(a) , where Q'(a) contains only positive labels, and each label in Q'(a), is congruent modulo p with a corresponding label in Q(a).

Now we formulate ***the Q(a)-Balanced Edge-Magic Graph Labeling Problem*** as follows.

INSTANCE: Given Graph G = (V,E), $|V| = p, |E| = q$, weights $w(v) \in Z^+$ be vertex sum for each $v \in V$ and $l(e) \in Q'(a) \subset Z^+$ for each $e \in E$, positive integers $K' = \sum_{x \in Q(a)'} x$, (i.e., the maximum vertex sums) and $J = \sum_{x \in Q(a)'} x$.

QUESTION: Is there a partition of V into disjoint $V_1, V_2, \ldots, V_p$ such that $w(v) = k + pm \leq K'$, for $v \in V_i$, $1 \leq i \leq p$, where m, $0 \leq k \leq p-1$ are integers such that $\sum_{e \in E} l(e) \leq J$?

By the above formulation, we could perform ***Polynomial Transformation*** from the Graph Partitioning Problem, which is NP-Complete shown in [16, p209], into the Q(a)-Balanced Edge-Magic Graph Labeling Problem. Thus, the Q(a)-Balanced Edge-Magic Graph Labeling Problem is NP-hard. "

Appendix

A NP-COMPLETE PROBLEM - GRAPH PARTITIONING [16, p209]

INSTANCE: Graph G = (V,E), weights $w(v) \in Z^+$, for each $v \in V$ and $l(e) \in Z^+$ for each $e \in E$, positive integers K and J.

QUESTION: Is there a partition of V into disjoint $V_1, V_2, \ldots, V_m$ such that $\sum_{v \in V_i} w(v) \leq K$ for $1 \leq i \leq m$ and such that if $E' \subseteq E$ is the set of edges that have their two end points in two different sets V_i then $\sum_{e \in E'} l(e) \leq J$?

Comment: Remains NP-complete for fixed $K \geq 3$ even if all vertex and edges weights are 1. Can be solved in polynomial time for $K=2$ by matching.

References

1. Dharam Chopra, H. Kwong and Sin-Min Lee, On the edge-magic (p,3p-1)-graphs, *Congressus Numerantium*, **179** (2006), 49-63.

2. Dharam Chopra, R. Dios and Sin-Min Lee, On The edge-magicness of zykov sum of graphs, to appear in *JCMCC*.

3. J.A. Gallian, A dynamic survey of graph labeling, *The Electronic J. of Combin.* (2001), # **DS6**, 1-144.

4. Sin-Min Lee, H.K. Ng and Yihui Wen, On the edge-Magic indices of (v, v+1)-graphs, to appear in *Utilita Mathematica*.

5. Sin-Min Lee, W.M. Pigg and T.J. Cox, On edge-magic cubic graphs conjecture, *Congressus Numerantium*, **105**, 214-222,1994.

6. Sin Min Lee, Eric Seah and S.K. Tan, On edge-magic graphs, *Congressus Numerantium*, **86**(1992), 179-191.

7. Sin-Min Lee, E. Seah, Siu-Ming Tong, On the edge-magic and edge-graceful total graphs conjecture, *Congressus Numerantium* **141**, 37-48 ,1999.

8. S. M. Lee, M. Kitagaki, J. Young, and W. Kocay, On Edge-graceful and edge-magic maximal outerplanar graphs, J. Combin. Math. Combin. Comput., **59** (2006) 119-129.

9. Sin-Min Lee, Wang Ling, and Yihui Wen, On Edge-magic cubic multi graphs and graphs, *Congressus Numerantium* **165**(2003), 145-160.

10. Sin-Min Lee, Thomas Wong, and Bill Lo, On Q(a)-balance edge magic graphs and Q(a)- balance super edge-magic graphs, Congressus Numerantium **188** (2007), 33-57.

11. Karl Schaffer and Sin Min Lee, Edge-graceful and edge-magic labelings of cartesian products of graphs, *Congressus Numerantium*, **141** (1999), 119-134,

12. W.C. Shiu, P.C.B. Lam and Sin-Min Lee, Edge-magicness of the composition of a cycle with a null graph, *Congressus Numerantium*, **132(**1998), 9-18.

13. W.C. Shiu, F.C.B. Lam and Sin-Min Lee, Edge-magic index sets of (p,p)-graphs, *Congressus Numerantium* **137**(1999), p.97-107.

14. W.C. Shiu, F.C.B. Lam and Sin-Min Lee, Edge-magic indices of multigraphs containing one more vertices than edges, manuscript.

15. W.C. Shiu and Sin-Min Lee, Some edge-magic cubic graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing,* **40** (2002),115-127.

16. M. R. Garey and D. S. Johnson, COMPUTERS AND INTRACTABILITY-A guide to the Theory of NP-Completeness, W.H. Freeman and Company, New York, 1979.