HOMEWORK PROBLEMS ABOUT THE GAUSS MAP

Composing the surface with its Gauss map, we obtain a map from the parameter domain to the sphere. This is also often referred to as the Gauss map. The “Gaussian image” is the range of the Gauss map. The “Gaussian area” is the area of the Gaussian image.

1. Calculate the Gauss map of the catenoid that you worked with in a previous homework assignment. There, you found that there is a stable and an unstable catenoid for some boundaries. Calculate the Gaussian image of both and show that the Gaussian area of the stable catenoid is less than the area of a hemisphere while that of the unstable catenoid is more than the area of a hemisphere. (That is not an accident, as we may have time to see later in the semester, if there is time to define and discuss stability.)

2. Consider the Gauss map of Ennper’s surface, which you already computed in an earlier assignment. Review your work on that assignment to find out for which value of $r$ the Gaussian image of the disk of radius $r$ in the parameter domain is exactly a hemisphere. Generalize the work you did on that assignment to prove the following theorem: Suppose that $M$ is any $C^2$ surface whose Gaussian image lies in the upper hemisphere. Then $M$ can be written in nonparametric form, i.e. in the form $Z = f(X, Y)$.

3. Consider the Gauss map of the helicoid. Which portions of the helicoid have their Gaussian image with area less than that of a hemisphere? Stability means that nearby surfaces with the same boundary have larger area. Intuitively, that means that if the surface is realized in soap film, it won’t just pop or move immediately to another surface. Which portions of the helicoid do you think are stable and what are their Gaussian images? (There is a theorem that having Gaussian area less than a hemisphere implies stability, but the converse is false.) I realize this last question is a bit vague but think about it anyway.