On the Supermagic Edge-splitting Extension of Graphs

Yihui Wen
Department of Mathematics
Suzhou Science and Technology College
Suzhou, Jiangsu 215009
People's Republic of China

Sin-Min Lee
Department of Computer Science
San Jose State University
San Jose, California 95192 U.S.A.

Hugo Sun
Department of Mathematics
California State University
Fresno, California 93720 U.S.A.

ABSTRACT. A \((p,q)\)-graph \(G\) in which the edges are labeled \(1,2,3,...,q\) so that the vertex sums are constant, is called supermagic. If the vertex sum mod \(p\) is a constant, then \(G\) is called edge-magic. We investigate the supermagic characteristic of a simple graph \(G\), and its edge-splitting extension \(SPE(G,l)\). The construction provides an abundance of new supermagic multigraphs.

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1. Introduction. Magic graphs were first initiated by Sedlacek (around 1963) [14,15] as the problem of labeling the edges of the graph with real numbers so that the sum of the edges label to a vertex is same as all the other vertices. Jeurissen [5], Jezney and Trenkler [6] gave characterizations of magic graphs. A characterization of regular magic graphs in term of even circuits is given by Doob [1]. Kong, Sun, Lee et al [7,24,25] provided some general constructions of magic graphs. Since Sedlacek's original article [14], literally hundred of papers have been written on magic graph labelings (see the survey article [2]).

If \(G\) is a \((p,q)\)-graph in which the edges are labeled \(1,2,3,...,q\) so that the vertex sums defined by \(f^+(u) = \Sigma f(u,v)\): \((u,v)\) in \(E\) is constant, then \(G\) is called supermagic. Figure 1 shows a graph with 6 vertices and 8 edges which is supermagic.

e-mail addresses:
1)yihuiwen6@sohu.com 2)lee@cs.sjsu.edu 3)hugo_sun@csufresno.edu

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Figure 1

B.M. Stewart [22, 23] showed that $K_3$, $K_4$, $K_5$ are not supermagic and when $n = 0 \pmod{4}$, $K_n$ is not supermagic. For $n > 5$, $K_n$ is supermagic if and only if $n = 0 \pmod{4}$. Hartsfield and G. Ringel [2] provided some classes of supermagic graphs. Ho and Lee [4] extended the result of Stewart to regular complete $k$-partite graphs. Recently Shiu, Lam and Cheng [17] considered a class of supermagic graphs which are disjoint union of $K_{3,3}$.

A generalization of supermagic graphs was introduced by Lee, Seah and Tan [10]. A graph $G = (V, E)$ with $p$ vertices and $q$ edges is called edge-magic if there is a bijection $f : E \rightarrow \{1, 2, \ldots, q\}$ such that the induced mapping $f^+ : V \rightarrow Z_p$, given by $f^+(u) = \sum f(u, v) : (u, v) \in E \pmod{p}$ is a constant. A necessary condition for a $(p, q)$-graph to be edge-magic is $q(q+1) = 0 \pmod{p}$. However, there are infinitely many connected graphs such as trees, cycles satisfy the necessary condition but are not edge-magic.

Figure 2

The concept of edge-magic labeling of graphs is the dual concept of edge-graceful labeling [10]. In 1985, Lo Sheng-Ping [13] introduced the concept of edge-graceful graphs. A $(p, q)$-graph $G = (V, E)$, of $p$ vertices and $q$ edges, is said to be edge-graceful if there exists a bijection $f : E \rightarrow \{1, 2, \ldots, q\}$ such that the induced mapping $f^+ : V \rightarrow \{0, 1, \ldots, p-1\}$, defined by $f^+(v) = \sum f(u, v) : (u, v) \in E(G) \pmod{p}$ is a bijection.
The cartesian product of two paths is frequently called the grid graph. The cartesian product of two cycles is called the torus graph. It was shown in [16] that the torus graph $C_m \times C_n$ is edge-magic for all $m,n \geq 2$.

Karl Schaffer and Sin-Min Lee [16] have shown that if $G$ and $H$ are both odd-order, regular, edge-graceful graphs, where $G$ is $d$-regular and has $m$ vertices, and $H$ is $k$-regular and has $n$ vertices, and furthermore $\text{GCD}(d,n) = \text{GCD}(k,m) = 1$, then $G \times H$ is edge-graceful. In particular, they showed that the torus graph $C_{2i+1} \times C_{2j+1}$ is edge-graceful.

Finding the magic labeling of graphs are related to solving system of linear Diophantine equations [21]. In general it is difficult to find an edge-magic or supermagic labeling of a graph. Several classes of graphs had been shown to be edge-magic ( [8,9,10,11,12,17,18] ). For more conjectures and open problems on edge-magic graphs the reader is referred to [8,9,10,11,12].

Shiu, Lam and Lee [18,19] give a general construction of supermagic graphs and edge-magic graphs. The reader should see the survey article of Gallian [2] for various labeling problems.

In this paper we introduce a construction of super magic graphs by splitting some of the edges of the graphs. We consider this construction for perfect matchings and cycles.

2. Edge-splitting extensions of graphs.

In this section, we shall introduce a general construction of multigraphs from a given simple graph. Let $N=\{1,2,3,\ldots\}$ be a set of natural numbers.

Given a pair $(G, f)$ where $G = (V,E)$ is a simple graph with $p$ vertices and $q$ edges and $f: E(G) \to N$, we can construct a graph $\text{SPE}(G,f)$ as follows:

For each edge $e$ of $E(G)$ if $f(e)=k$, we associate a set of parallel edges $P(e) = \{e_1, e_2, \ldots, e_k\}$. We observe that $V(\text{SPE}(G,f)) = V(G)$ and $E(\text{SPE}(G,f)) = \cup \{P(e): e \in E(G)\}$

We shall call the graph $\text{SPE}(G,f)$ as an edge-splitting extension graph of $(G,f)$.

We illustrate here with one example:

**Example 1.** Let $G = C_4$ and $f: E(C_4) \to N$ as follows:

![Figure 3](image-url)
A necessary condition for a \((p,q)\)-graph to be supermagic is \(q(q+1) \equiv 0 \pmod{p}\). However, this is not the sufficient condition. Consider the following example:

**Example 2.** Let \(G = C_6\) and \(h : E(C_6) \to \mathbb{N}\) be defined as follows:

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>h</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Then we see that the following SPE\((C_6,h)\) (Figure 4) has 6 vertices and 9 edges.

![Figure 4](image)

If the graph is supermagic then three single edges must be labeled by \(x\) for some \(x\) which is not possible.

**Remark.** In fact, if \(G\) is a graph with three edges \(a, b,\) and \(c\) such that \(a\) is adjacent to \(b\) and \(b\) is adjacent to \(c\) but \(a\) and \(c\) are not adjacent and \(f : E(G) \to \mathbb{N}\) with property \(f(a) = f(c) = 1\), then SPE\((G,f)\) is not supermagic.

**Theorem 1.** If \(G\) is a regular simple graph and \(f : E(G) \to \mathbb{N}\) is a constant map with \(f(e) = 2t\) for a fixed \(t\), then SPE\((G,f)\) is supermagic.

**Proof.** Suppose \(G\) is a \((p,q)\)-graph. We see that SPE\((G,f)\) has \(2tq\) edges. We divide the set of integers \(\{1, 2, \ldots, 2tq\}\) into \(tq\) pairs \(\{1, 2tq\}, \{2, 2tq-1\}, \ldots, \{k, 2tq-k+1\}, \ldots, \{tq, tq+1\}\). We denote this set by \(H\).

For each \(e\) of \(G\), \(P(e) = \{e_1, e_2, \ldots, e_{2t}\}\) we form the \(tq\) pairs \(\{e_1, e_2\}, \{e_3, e_4\}, \ldots, \{e_{2t-1}, e_{2t}\}\).

Then the set of edges \(\text{SPE}(G,f)\) has \(tq\) pairs. We denote the set by \(Q\). Any bijection \(F : Q \to H\) induces a super edge-magic labeling \(g : E(\text{SPE}(G,f)) \to \{1, 2, \ldots, 2tq\}\) by letting \(g(a) = i\) and \(g(b) = j\) if \(F(e_i, e_j) = (i, j)\).

There exists graph \(G\) such that \(\text{SPE}(G,f)\) is not supermagic for any \(f : E(G) \to \mathbb{N}\). For example a connected graph \(G\) has a tail of length greater than or equal to one has this property. We can formulate the above observation by the following result.

**Theorem 2.** Any graph \(G\) with two vertices \(u, v\) such that \(\text{deg}(u) = 1\) and \(\text{deg}(v) \geq 2\) and \(u, v\) are adjacent has the property that \(\text{SPE}(G, f)\) is not supermagic for any \(f\) \(E(G) \to \mathbb{N}\).

**Corollary 3.** For any \(n > 1\), the complete bipartite graph \(K_{1,n}\) has the property that \(\text{SPE}(K_{1,n}, f)\) is not supermagic for any \(f\) \(E(G) \to \mathbb{N}\).
Corollary 4. If n > 2, then the graph SEP(P_n, f) is not supermagic for all f: E(G) → N.

Theorem 5(a) The graph G = P_2 has the property that SPE(G, f) is supermagic for any f: E(G) → N. 

(b) P_2 is the only graph G with the property that SPE(G, f) is supermagic for any f: E(G) → N. 

3. Edge-splitting extension of perfect matchings.

Let mK_2 be the m perfect matching and f: E(mK_2) → N with f(e) = n for all e in E(G).

Shiu and Lee [20] showed that:

Theorem 6. For m, n ≥ 2, the splitting-edge extension of the m perfect matching and f: E(G) → N with f(e) = n for all e in E(G) is supermagic if and only if n is even or both m and n are odd.

Example 3. We label SPE(3K_2, f) where f: E(3K_2) → N with f(e) = 3 for all e of 3K_2 as follows:

Figure 5

Theorem 7. If the graph SPE(2K_2, f) where f: E(2K_2) → N is supermagic, then Σf(e)≡1 (mod 4) or Σf(e)≡2 (mod 4).

Proof. If Σf(e)=1 (mod 4) then Σf(e)=4k+1 for some k. As q=4k+1, we have q(q+1) = (4k+1)(4k+2) = 2(4k+1)(2k+1).

Now p=4. It is clear that q(q+1) ≡ 0 (mod 4). Thus SPE(2K_2, f) cannot be supermagic.

If Σf(e)=2 (mod 4) then Σf(e)=4k+2 for some k. As q = 4k+2, we have q(q+1) = (4k+2)(4k+3) = 2(4k+1)(2k+1). Now p=4. It is clear that q(q+1)≡ 0 (mod 4). Thus SPE(2K_2, f) cannot be supermagic. 

Theorem 8. The graph SPE(2K_2, f) where f: E(2K_2) → N is supermagic if f(a)=2k-1, f(b)=2k for k≥1.

Proof. We see that SPE(2K_2, f) has 4k-1 edges. We label the edges P(a) = {a_1, a_2, ..., a_{2k-1}} by A = {1, 2, 3, ..., [(2k-1)/2], 4k-1, 4k-2, ..., 4k-[(2k-1)/2]-1}. The other edges P(b) = {b_1, b_2, ..., b_{2k}} will be labeled by the complement of A in {1, 2, ..., 4k-1}.

We observe that this is a supermagic labeling. 

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Example 4. We can have a supermagic labeling of $\text{SPE}(2K_2, f)$ with $f(a)=7, f(b)=8$ as follows:

\{1,2,3, 15,14,13,12\}, \{4,5,6,7,8,9,10,11\}.

Theorem 9. The graph $\text{SPE}(2K_2, f)$, where $f : E(2K_2, f) \rightarrow N$, is supermagic if $f(e_1) = n, f(e_2) = 2n$ for $n \equiv 0 \pmod{4}$ or $n \equiv 1 \pmod{4}$.

Proof. We consider two cases:

Case 1. $n \equiv 0 \pmod{4}$. Let $n = 4k, k \geq 1$.

The sum of the first $8k$ integers is $4k(8k+1)$, and the sum of the last $4k$ integers is $2k(20k+1)$. Since they differ by $8k^2 - 2k = (k-1)8k + 6k$. We switch $\{7k + 2, 7k + 3, ..., 8k\}$ with $\{11k + 2, 11k + 3, ..., 12k\}$ and $10k$ with $7k$. For the $k-1$ pairs of swap, the first sum increases by $4k$ and the second sum decreases by $4k$, while the last pair increases the first sum by $3k$ and decreases the second sum by $3k$. As a result, they both sum up to $3k(12k + 1) = 3n(3n + 1)/4$. Therefore $\text{SPE}(2K_2, f)$ is supermagic.

Case 2. $n \equiv 1 \pmod{4}$. Let $n = 4k+1, k \geq 1$.

The sum of the first $8k + 2$ numbers is $(4k + 1)(8k + 3)$, and the last $4k + 1$ numbers sum up to $(4k + 1)(10k + 3)$. They differ by $2k(4k + 1)$; switching the last $k$ numbers will give both sums $(4k + 1)(9k + 3)$. We see that $\text{SPE}(2K_2, f)$ is supermagic.

Example 5. (a) If $\text{SPE}(2K_2, f)$ has $f(e_1) = 8$ and $f(e_2) = 16$, we follow case 1 of Theorem 10, with $k = 2$ and label $\text{P}(e_1) = \{e_{1,1}, e_{1,2}, ..., e_{1,8}\}$ by $\{17, 18, ..., 24\}$ $\text{P}(e_2) = \{e_{2,1}, e_{2,2}, ..., e_{2,16}\}$ by $\{1, 2, ..., 16\}$, switching 24 with 16 and 20 with 14, we have a supermagic labeling.

(b) If $\text{SPE}(2K_2, f)$ has $f(e_1) = 13$ and $f(e_2) = 26$, we follow case 2 of Theorem 10, with $k = 3$ and label $\text{P}(e_1) = \{e_{1,1}, e_{1,2}, ..., e_{1,13}\}$ by $\{27, 28, ..., 39\}$ $\text{P}(e_2) = \{e_{2,1}, e_{2,2}, ..., e_{2,26}\}$ by $\{1, 2, ..., 26\}$, switching 39 with 26, 38 with 25, and 37 with 24, we have a supermagic labeling.

Remark. When $m = 3$ and $f : E(3K_2) \rightarrow N$ satisfies the necessary condition of supermagicness, that is $(\Sigma f(e)) (1+\Sigma f(e)) \equiv 0 \pmod{6}$, we do not guarantee that $\text{SPE}(3K_2, f)$ is supermagic.

For example if $E(3K_2) = \{a,b,c\}$ and we consider the following two mappings:

(1) $f(a)=1, f(b)=2$ and $f(c)=3$. We see that $5x6 \equiv 0 \pmod{6}$. The graph $\text{SPE}(3K_2, f)$ is supermagic with the label \{5\}, \{1,4\} and \{2,3\}.

(2) $f(a)=1, f(b)=2$ and $f(c)=3$. We see that $6x7 \equiv 0 \pmod{6}$. If $\text{SPE}(3K_2,f)$ is supermagic then the vertex sum should be 7 which is not possible to have such a labeling.

Theorem 10 A necessary condition for $\text{SPE}(3K_2, f)$ with $f : E(3K_2) \rightarrow N$, $f(e_1) = f(e_2) = n, f(e_3) = k$ to be supermagic is $q(q+1) \equiv 0 \pmod{6}$ and $(q+1)/3k \leq 2q+1 - k$. 120
Proof. Suppose \( f(e_1) = f(e_2) = n, f(e_3) = k \) and \( \text{SPE}(3K_2, f) \) is supermagic. Let \( A = \{x_1, x_2, \ldots, x_n\}, B = \{y_1, y_2, \ldots, y_n\}, C = \{z_1, z_2, \ldots, z_k\} \) be the edge labels of \( \text{SPE}(3K_2, f) \). Then we have

\[
\sum x_i = \sum y_j = \sum z_m = q(q+1)/6.
\]

Since the sum of \( \{q, (q-1), \ldots, q-k+1\} \) is \((k/2)(2q+1-k)\), thus \( q(q+1)/6 \leq (k/2)(2q+1-k) \), i.e., \( q(q+1)/3k \leq 2q+1-k \). \( \square \)

**Remark.** The above necessary condition for \( \text{SPE}(3K_2, f) \) to be supermagic is not sufficient in general.

Consider \( \text{SPE}(3K_2, f) \) with \( f(e_1) = f(e_2) = 2, f(e_3) = 7 \). Here \( q = 11 \) and \( q(q+1)/6 = 22 \). We observe that \( q(q+1)/3k \leq 2q+1-k \).

However, it is impossible to find two numbers from \( \{1, 2, \ldots, 11\} \) whose sum is 22; hence, \( \text{SPE}(3K_2, f) \) is not supermagic. \( \square \)

**Theorem 11.** \( \text{SPE}(3K_2, f) \) is supermagic if \( f : E(3K_2) \rightarrow \mathbb{N} \) is \( f(e_1) = f(e_2) = n, f(e_3) = 2n \), for \( n \equiv 0 \pmod{3} \) or \( n \equiv 2 \pmod{3} \).

**Proof.** Case 1. \( n \equiv 0 \pmod{3} \). Let \( n = 3k \). Break the set \( \{1, 2, \ldots, 12k\} \) into 3 piles: \( \{1, 2, \ldots, 6k\}, \{6k+1, 6k+2, \ldots, 9k\} \), and \( \{9k+1, 9k+2, \ldots, 12k\} \). The first pile sums up to \( 3k(6k+1) = 18k^2 + 3k \); the second pile sums up to \( (3k/2)(15k+1) = (1/2)(45k^2+3k) \); and the third pile sums up to \( (3k/2)(21k+1) = (1/2)(63k^2+3k) \).

Switch \( k \) numbers in the first pile with \( k \) numbers in the third pile with pairwise difference \( 6k-1 \); e.g., \( \{4k+2, 4k+3, \ldots, 5k+1\} \) with \( \{10k+1, 10k+2, \ldots, 11k\} \).

Next if \( k \) is odd, we switch \( k \) numbers in the second pile with \( k \) number in the third pile with pairwise difference \( (3k+1)/2 \); e.g., \( \{7k+(k+1)/2, 7k+(k+3)/2, \ldots, 8k+(k-1)/2\} \) with \( \{9k+1, 9k+2, \ldots, 10k\} \).

If \( k \) is even, we simply switch any \( k/2 \) numbers of the second pile with the \( k/2 \) numbers of the third pile which differ pairwise by \( 3k+1 \); e.g., \( \{8k+(k/2), \ldots, 9k-1\} \) with \( \{11k+(k/2)+1, \ldots, 12k\} \). Now all three piles sum up to \( 24k^2+2k \).

Case 2. For \( n \equiv 2 \pmod{3} \), let \( n = 3k+2 \). We break the set \( \{1, 2, \ldots, 12k+8\} \) into three piles as before, \( \{1, 2, \ldots, 6k+4\}, \{6k+5, 6k+6, \ldots, 9k+6\}, \{9k+7, \ldots, 12k+8\} \).

The first pile sums up to \( (3k+2)(6k+5) = 18k^2 + 27k + 10 \), the second pile sums up to \( (1/2)(3k+2)(15k+1) \), and the third pile sums up to \( (1/2)(3k+2)(21k+15) \).

We first switch \( k \) numbers in the first pile with the \( k \) numbers in the third pile with pairwise difference \( 6k \), and an extra pair with difference \( 7k + 2 \); e.g., \( \{5k+5, \ldots, 6k+4; 2k+5\} \) with \( \{11k+5, \ldots, 12k+4; 9k+7\} \).

If \( k \) is odd, we switch \((k+1)/2\) numbers in the second pile with the \((k+1)/2\) numbers in the third pile with pairwise difference \( 3k+2 \); e.g., \( \{7k+1, \ldots, 7k+(k+1)/2\} \) with \( \{10k+3, 10k+4, \ldots, 10k+(k+1)/2 + 2\} \).

If \( k \) is even, we simply switch any \( k/2 \) numbers in the second pile with the \( k/2 \) numbers in the third pile with pairwise difference \( 3k \) and one pair with difference \( (5k+2)/2 \).

Then all three piles sum up to \( (6k+4)(4k+3) \). \( \square \)
Example 6. If $n = 14$, $f(e_1) = f(e_2) = 14$, and $f(e_3) = 28$, we label $e_1$ with \{29, 30, \ldots, 37, 45, 46, \ldots, 49\}, $e_2$ with \{1, 26, 27, 28, 38, 39, 40, 41, 42, 44, 45, 50, 51, 52, 53\}, and $e_3$ with \{2, 3, \ldots, 25, 43, 54, 55, 56\}. 
Now all three piles sum up to 532.

Remark. SPE(3K₂, f) is not supermagic if $f : E(C_{3k+1}) \to \mathbb{N}$ is $f(e_1) = f(e_2) = n$, $f(e_3) = 2n$, and $n \equiv 1 \pmod{3}$. For if $n = 3k+1$, then as $q = 4n = 12k+4$. We see that the $q(q+1) = 4(3k+1)(12k+5) \neq 0 \pmod{6}$. Therefore it is not supermagic.

4. Edge-splitting extensions of cycles

In this section we investigate supermagicness of edge-splitting extensions of cycles.

**Theorem 12.** If $m$ is odd and $k \geq 1$, then the cycle $C_{3m}$ has supermagic edge splitting extension for all $f : E(C_{3m}) \to \mathbb{N}$ with $f(e) = n$ for all $e$ in $C_{3m}$.

For even $n$, Theorem 12 is a corollary of Theorem 1. For odd $n$, it is sufficient to prove for $n = 3$, a simple induction will prove for all odd $n > 3$.

We need the following Lemmas in the system of Diophantine equations:

**Lemma 4.1.** Let $m$ be odd and let $s = 3(3m+1)/2$. A system of $m$ Diophantine equations

\[
X_{1,1} + X_{2,1} + X_{3,1} = s \\
X_{1,2} + X_{2,2} + X_{3,2} = s \\
X_{1,3} + X_{2,3} + X_{3,3} = s
\]

\[
X_{1,m} + X_{2,m} + X_{3,m} = s
\]

has a solution with distinct $X_{ij}$ in \{1, 2, \ldots, 3m\}.

**Proof.** Set $X_{1,1} = 1$, $X_{1,2} = 2$, $X_{1,3} = 3$, \ldots, $X_{1,k} = k$, \ldots, $X_{1,m} = m$; $X_{2,m} = m+1$, $X_{2,m-2} = m+2$, \ldots, $X_{2,m-k} = m+1+k$, \ldots, $X_{2,1} = m+(m+1)/2$ and $X_{2,m-1} = m+(m+3)/2$. Thus $X_{2,m} = m+(m+1)/2, X_{2,m-1} = 2m, X_{2,m-2} = 2m+(m+3)/2, \ldots, X_{2,1} = 2m$.

The sums in each row are now distinct consecutive integers $m+(m+3)/2, m+(m+5)/2, \ldots, 2m+(m+1)/2$.

Thus setting $X_{3,n-1} = 2m+1$, $X_{3,m-2} = 2m+2$, \ldots, $X_{3,2} = 2m+(m-1)/2, X_{3,m} = 2m+(m+1)/2, X_{3,m-1} = 2m+(m+3)/2, \ldots, X_{3,1} = 3m$.

We see that the sum of each line yields $4m+(m+3)/2 = 3(3m+1)/2$. \[\]

**Lemma 4.2.** Let $m \geq 4$ be even, and let $s = (9m+2)/2$. Then the system of $m$ Diophantine equations

\[
X_{1,1} + X_{2,1} + X_{3,1} = s \\
X_{1,2} + X_{2,2} + X_{3,2} = s+1 \\
X_{1,3} + X_{2,3} + X_{3,3} = s
\]

\[
X_{1,m} + X_{2,m} + X_{3,m} = s+1
\]

has a solution in distinct integers in \{1, 2, \ldots, 3m\}.

**Proof.** Set $X_{1,1} = 1$, $X_{1,2} = 2$, $X_{1,3} = 3$, \ldots, $X_{1,k} = k$, \ldots, $X_{1,m} = m$; $X_{2,m-1} = m+1$, $X_{2,m-3} = m+2$, \ldots, $X_{2,1} = m+(m/2)$, $X_{2,m} = m+(m/2)+1$, $X_{2,m-2} = m+(m/2)+2$, \ldots, $X_{2,2} = 2m$. 

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The sums of each row are now $m+ (m/2)+1, m+ (m/2)+2, \ldots, 2m+ (m/2)$. They are consecutive and distinct, except $2m+1$ does not occur. By setting $X_{3,m} = 2m+1, X_{3,m-2} = 2m+2, \ldots, X_{3,2} = 2m+ (m/2); X_{3,m-1} = 2m+ (m/2)+1, X_{3,m-3} = 2m+ (m/2)+2, \ldots, X_{3,1} = 3m$.

We see that the odd rows sum up to be $4m+ (m/2)+1 = s$, and the even rows $4m+ (m/2)+2 = s+1$. □

Proof of Theorem 12. Let $E(C_m) = \{e_1, e_2, \ldots, e_m\}$ and for each $k$ between 1 and $m$, let $\{e_{k,1}, e_{k,2}, \ldots, e_{k,n}\}$ be the parallel edges of $SPE(C_m, f)$ correspond to $e_k$.

We define the edge labeling $g$: $SPE(C_m, f) \rightarrow \{1, 2, \ldots, mn\}$ by setting $g( e_{kj} ) = X_{kj}$ for each $k$ between 1 and $m$ and $j$ between 1 and 3.

Then by Lemma 4.1 and Lemma 4.2, it is a supermagic labeling.

For $n \geq 5$, we label the edges $3m+1, 3m+2, \ldots, 4m$ consecutively in the counterclockwise manner; then label the next cycle $4m+1, 4m+2, \ldots, 5m$ in the clockwise manner, with the edge labels $4m$ and $4m+1$ having the same vertices.

This process is repeated until all the edges are labeled.

It is easy to see the labeling is supermagic.

Example 7. We give a supermagic labeling of $SPE(C_7, f_3)$. We set $X_{1,1} = 1, X_{1,2} = 2, X_{1,3} = 3, X_{1,4} = 4, X_{1,5} = 5, X_{1,6} = 6, X_{1,7} = 7, X_{2,1} = 11, X_{2,2} = 10, X_{2,5} = 9, X_{2,7} = 8, X_{2,2} = 14, X_{2,4} = 13, X_{2,6} = 12$.

Then we set $X_{3,2} = 17, X_{3,4} = 16, X_{3,6} = 15, X_{3,1} = 21, X_{3,3} = 20, X_{3,5} = 19, X_{3,7} = 18$, then we observe that each vertex has sum $2s = 2 \times 33 = 66$ (see Figure 6):

*Figure 6*

Example 8. We give a supermagic labeling of $SPE(C_5, f_5)$, we see that
Example 9. Let $G = C_3$ and $f_i : E(C_3) \to \mathbb{N}$ for $i = 1, 2$ be defined as follows:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1$</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>$f_2$</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Then we see that $\text{SPE}(C_3, f_1)$ is not supermagic. However, $\text{SEP}(C_3, f_2)$ is supermagic (see Figure 8).

![Figure 8](image)

$\text{SPE}(C_3, f_1)$ is not supermagic \hspace{1cm} $\text{SPE}(C_3, f_2)$ is supermagic

Theorem 13. Suppose $f(a) = f(b) = n$ and $f(c) = 1$ in $C_3$ then $\text{SPE}(C_3, f)$ is supermagic if and only if $n = 2$.

Theorem 14. Suppose $f(a) = f(b) = n$ and $f(c) = 2$ in $C_3$, then $\text{SPE}(C_3, f)$ is supermagic if and only if $n = 2$ or $3$.

Proof. If $n = 2$ or $3$, the supermagicness of $\text{SPE}(C_3, f)$ is shown as Figure 9.
Theorem 15. Suppose \( f(e_1) = k, f(e_2) = k+1 \) and \( f(e_3) = k+2 \) in \( C_3 \), then \( \text{SPE}(C_3, f) \) is (a) not super magic for \( k = 1 \). (b) supermagic for \( k \geq 2 \).

Proof. (a) If \( k = 1 \), \( \text{SPE}(C_3, f) \) has the following configuration (see Figure 10):

![Figure 10](image)

We see that \( a = b + c = d + e + f \) or \( a + b + c = a + d + e + f = b + c + d + e + f \) has no solution in \( \{1, 2, 3, 4, 5, 6\} \).

The only way to have \( a = d + e + f \) is \( a = 6 \), \( \{d, e, f\} = \{1, 2, 3\} \) but there is no way to obtain \( b \) and \( c \).

(b) Assume now that \( k \geq 2 \), we see that \( q(q+1) = 3(n+1)(3n+4) = 0 \) (mod 3).

Now we set

\[
C_1 = \{ x: x \equiv 4 \pmod{6} \text{ or } x \equiv 3 \pmod{6}, 9 < x \leq q \}
\]

\[
C_2 = \{ y: y \equiv 2 \pmod{3}, 9 < y \leq q \}
\]

\[
C_3 = \{ z: z \equiv 0 \pmod{6} \text{ or } z \equiv 1 \pmod{6}, 9 < z \leq q \}
\]

If \( k \) is even and \( k = 2m \), \( f(e_1) = k, f(e_2) = k + 1 \) and \( f(e_3) = k + 2 \).

Let \( A_1 = \{6, 9\} \cup C_1, A_2 = \{2, 5, 8\} \cup C_2, A_3 = \{1, 3, 4, 7\} \cup C_3 \) be the edge labels for parallel edges of \( e_1, e_2, e_3 \), respectively.

For any \( u \) in \( C_3 \) we have \( f'(u) = 2(3m+2)(2m+1) \).

If \( k \) is odd \( k = 2m+1 \), we set \( B_1 = \{7, 9\} \cup C_1, B_2 = \{2, 5, 8\} \cup C_2, B_3 = \{1, 3, 4, 6\} \cup C_3 \) as the edge labels for parallel edges of \( e_1, e_2, e_3 \) respectively.

For all \( u \) in \( C \) we have \( f'(u) = 2(m+1)(6m+7) \).

Thus when \( k \geq 2 \), \( \text{SPE}(C_3, f) \) is supermagic. \( \square \)

Example 10. We label \( \text{SPE}(C_3, f) \) for \( k = 2 \) as follows: \( \{7, 8\}, \{2, 4, 9\}, \{1, 3, 5, 6\} \);
for $k = 3$ as follows: $\{7,9,10\}$, $\{1,2,11,12\}$, $\{3,4,5,6,8\}$ for $k = 4$ as follows: $\{1,10,14,15\}$, $\{4,5,6,12,13\}$, $\{2,3,7,8,9,10\}$.

If we use the method of Theorem 16 we can have another super magic labeling:

$A_1 = \{6,9,10,15\}$, $A_2 = \{2,5,8,11,14\}$ and $A_3 = \{1,3,4,7,12,13\}$

For each $u$ in $C_3$, we have $f'(u) = 80$.

For $k = 5$, $f(e_i) = k$, $f(e_2) = k+1$, $f(e_3) = k+2$, in C, the labels of splitting edges of $e_1,e_2,e_3$ are

$A_1 = \{7,9,10,15,16\}$, $A_2 = \{2,5,8,11,14,17\}$ and $A_3 = \{1,3,4,6,12,13,18\}$

For each $u$ in $C_3$, we have $f'(u) = 114$.

When $n$ is even, we observe that if we define $f_3 : C_4 \to N$ by $f_3(e) = 3$ for all $e$ in $C_4$. The graph SPE($C_4$, $f$) has 12 edges. It satisfies the necessary supermagic condition: $q(q+1) \equiv 0 \pmod{p}$. It is possible for us to divide $\{1,2,\ldots,12\}$ into 4 group of triples $\{x_1,x_2,x_3\} : \{1,7,11\}$, $\{2,8,9\}$ with sum 19 and $\{3,5,12\}$, $\{4,6,10\}$ with sum 20. So if we label the splitting edges of $(v_1,v_2)$, $(v_2,v_3)$, $(v_3,v_4)$ and $(v_4,v_1)$ consecutively by $\{1,7,11\}$, $\{3,5,12\}$, $\{2,8,9\}$, $\{4,6,10\}$. Then we see that SPE($C_4$, $f$) is supermagic.

The result can be generalized to the following

**Theorem 16.** SPE($C_{2n}$, $f$) is supermagic if for any $t \geq 2$ $f(e_i) = t$ for all $e_i$ in $E(C_{2n})$.

**Proof.** (1) When $t = 2m$ by Theorem 1, we conclude that SPE($C_{2n}$, $f$) is supermagic,

(2) Assume $t = 2m+1$, in order to prove that SPE($C_{2n}$, $f$) is supermagic, it suffices to show that it is true for $t = 3$.

Since $f(e) = 3$, we have $q = \sum f(e) = 6n$, we want to partition the set $\{1,2,\ldots,6n\}$ into the following sets:

- $A(k) = \{k, 4n+k, 4n+2(k-1)\}$, $k = 1,2,\ldots,n$,
- $B(k) = \{n+k, 4n-(2k-1), 5n+1\}$, $k = 1,2,\ldots,n$.

The sum of numbers in $A(k)$ is $\sigma_1 = 8n + 2$, and the sum of numbers in $B(k)$ is $\sigma_2 = 10n + 1$.

Assume $E(C_{2n}) = \{e_1,e_2,\ldots,e_{2n}\}$ we label the splitting edges of $e_{2i}$ by $A(i)$ and splitting edges of $e_{2i+1}$ by $B(i)$ $i = 1,2,\ldots,n$, then

$f'(u) = \sigma_1 + \sigma_2 = 3(6n + 1).

Therefore SPE($C_{2n}$, $f$) is supermagic. []

**Example 11.** For SPE($C_{13}$, $f$) with $f(e) = 3$ for all $e$ in $C_{10}$, we have $q = 30$. We partition $\{1,2,\ldots,30\}$ into the following 3-elements subsets

- $A(1) = \{1,20,21\}$, $A(2) = \{2,18,22\}$, $A(3) = \{3,16,23\}$
- $A(4) = \{4,14,24\}$, $A(5) = \{5,12,25\}$;
- $B(1) = \{6,19,26\}$, $B(2) = \{7,17,27\}$, $B(3) = \{8,15,28\}$
- $B(4) = \{9,13,29\}$, $B(5) = \{10,11,30\}$.

If we label $E(SPE(C_{10}$, $f$)) according the method of the method proof of Theorem 16, then we see that $\sigma_1 = 42$, $\sigma_2 = 51$, $f'(u) = \sigma_1 + \sigma_2 = 114$. 

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References


