On The Integer-Magic Spectra of Two-vertex sum of Paths

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Abstract: For $k>0$, we call a graph $G=(V,E)$ as $Z_k$-magic if there exists a labeling $l: E(G) \rightarrow Z_k^*$ such that the induced vertex set labeling $l^*: V(G) \rightarrow Z_k$ defined by $l^*(v) = \sum l(u,v) : (u,v) \in E(G)$ is a constant map. We denote the set of all $k$ such that $G$ is $k$-magic by $\text{IM}(G)$. We call this set as the integer-magic spectrum of $G$. We investigate the integer-magic spectra for graphs which are two-vertex sum of paths.

1. Introduction. For any abelian group $A$, written additively we denote $A^* = A - \{0\}$. Any mapping $l: E(G) \rightarrow A^*$ is called a labeling. Given a labeling on edge set of $G$ we can induced a vertex set labeling $l^*: V(G) \rightarrow A$ as follows:

$$l^*(v) = \sum l(u,v) : (u,v) \in E(G)$$

A graph $G$ is known as $A$-magic if there is a labeling $l: E(G) \rightarrow A^*$ such that for each vertex $v$, the sum of the labels of the edges incident with $v$ are all equal to the same constant; i.e., $l^*(v) = c$ for some fixed $c$ in $A$. We will called $\langle G,l \rangle$ a $A$-magic graph. In general, a graph $G$ may admit more than one $A$-magic labelings.

We denote the class of all graphs (either simple or multiple graphs) by $\text{Gph}$. The class of all abelian groups by $\text{Ab}$. For each $A$ in $\text{Ab}$ we denote the class of all $A$-magic graphs by $A\text{MGp}$. When $A = Z$, the $Z$-magic graphs were considered in Stanley[26,27]; he pointed out that the theory of magic labelings can be put into the more general context of linear homogeneous diophantine equations [26]. When the group is $Z_k$, we shall refer to the $Z_k$-magic graph as $k$-magic. Graphs which are $k$-magic had been studied in [10, 12,13,14,15,18,19].

The original concept of $A$-magic graph is due to J. Sedlacek [23,24], who defined it to be a graph with real-valued edge labeling such that (i) distinct edges have distinct nonnegative labels, and (ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

Doob [1,2,3] also considered $A$-magic graphs where $A$ is an abelian group. Given the graph $G$, the problem of deciding whether $G$ admits a magic labeling is equivalent to the problem of deciding whether a set of linear


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homogeneous Diophantine equation has a solution. At present, given an abelian group, no general efficient algorithm is known for finding magic labelings for general graphs.

In this paper we use $\mathbb{N}$ to denote the set $\{1,2,3,\ldots\}$ and for each $k > 0$, we write the set $\{kx: x \in \mathbb{N}\}$ by $k\mathbb{N}$ and $\{k+x: x \in \mathbb{N}\}$ by $k+\mathbb{N}$. We will define the graph $G$ with a magic labeling $l: E(G) \to \mathbb{N}$ as $N$-magic. It is well-known that a graph $G$ is $N$-magic if and only if each edge of $G$ is contained in a 1-factor (a perfect matching) or a $\{1,2\}$-factor (see [9,22,30,31]). Reader refer to [4,5,6,7,8,10,12,24,25,28,29] for $N$-magic graphs. The $Z$-magic is weaker than $N$-magic. Figure 1 shows a graph which is $Z$-magic but not $N$-magic.

![Figure 1](image1)

For simplicity, we will consider $Z$-magic as 1-magic. Given a graph $G$, we denote the set of all $k > 0$ such that $G$ is $k$-magic by $IM(G)$. We call this set as integer-magic spectrum of $G$. We investigate these sets for general graphs in [12,16]. Figure 2 shows a graph $G$ whose $IM(G) = N\{2,3,4\}$.

![Figure 2](image2)

A special type of amalgamation of graphs called the two-vertex sum was considered in [11] by the first author.
Let $\Omega = \{(G_i, \{u_i, v_i\}) : i \in I\}$ be a class of graphs with two distinguished vertices $u_i$ and $v_i$. The two vertex-sum of $\Omega$ is the disjoint union of $G_i$ by identification all $u_i$ and identification all $v_i$, under the following edge identification:

1. for those vertices $(u_i, v_i)$ which are adjacent we identify the edges into one.
2. for those vertices $(u_i, v_i)$ with distance $d((u_i, v_i)) > 1$, we keep the paths between them.

We will denote this resulting graph by $\Sigma(\Omega)$.

**Example 1.** Figure 3 shows a two-vertex sum of three cycles.

![Figure 3](image)

In this paper, we will consider the integer-magic spectra of graphs that are constructed from the two-vertex sum of paths. In particular, we completely determine the integer-magic spectra of the generalized theta graphs.

### 2. Generalized Theta Graphs.

If $\Omega = \{(P_i, \{u_i, v_i\}) : i \in I\}$ is a collection of paths with two ends $u_i$ and $v_i$ then $\Sigma(\Omega)$ is the generalized theta graph. If $|I| = 2$, then $\Sigma(\Omega)$ is isomorphic to a cycle. Thus $IM(\Sigma(\Omega)) = N$. So we may assume $|I| > 2$.

By our definition of two vertex sum if there are more than two $P_2$ in $\Omega$, all these $P_2$ will identified in $\Sigma(\Omega)$. Thus we may assume in $\Omega$, there is at most one $P_2$.

We consider two cases.

(a) $|I|$ is even.

Suppose $\Omega = \{(P_i, \{u_i, v_i\}) : i \in I\}$ and $|I|$ is even and greater than 2. Let $#_o = \text{total number of } n_j \text{ which are odd}$ and $#e = \text{total number of } n_j \text{ which are even}$. 

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Theorem 2.1. Given $\Omega = \{(P_i, \{u_i, v_i\}) : i \in I\}$ with $|I|$ even, we have the integer magic spectrum $\mathbf{IM}(\Sigma(\Omega)) = N$, if $\#o$ is even.

Proof. The graph $\Sigma(\Omega)$ is eulerian with even number of edges. Low and Lee [21] showed that if a graph $G$ is eulerian of even size then it is $A$-magic for any abelian group $A$. In particular, $\Sigma(\Omega)$ is $k$-magic for all $k$.

Thus, we have $\mathbf{IM}(\Sigma(\Omega)) = N$. $\Box$

Example 2. Figure 4 illustrates the result for $\emptyset (P_2, P_4, P_5, P_5)$.

Theorem 2.2. Given $\Omega = \{(P_i, \{u_i, v_i\}) : i \in I\}$ with $|I|$ even, we have the integer magic spectrum $\mathbf{IM}(\Sigma(\Omega)) = 2N$, if $\#o$ is odd.

Proof. Let $u$ be a vertex of degree $|I|=2m$ in $\Sigma(\Omega)$. Suppose $\Sigma(\Omega)$ has a $k$-magic labeling.

Consider two cases:

Case 1. $\#o \geq \#e$.

Without lost of generality, we assume the $\#o$ paths of odd length are arranged first in $\Omega$ and the remaining are $\#e$ paths of even length.

Then we label the edges of $\Sigma(\Omega)$ corresponding to the edges of the $2m$ paths $P$ starting from the edge incident with $u$ by $a$ and $-a$ alternately from top to bottom.

Next, we label the edges of $\Sigma(\Omega)$ corresponding to the edges of the path $P_n$ starting from the edge incident with $u$ by $a$ and $-a$ alternately or $-a$ and $a$ alternately. If $n$ is even, the last edge incident on $v$ will have the same edge labels as the first edge incident on $u$. If $n$ is odd, the last edge incident on $v$ will have the inverse edge labels of the label of the first edge incident on $u$.

Then we observe the vertex $v$ of $\Sigma(\Omega)$ will have labels $-2a=0 \pmod{k}$. This shows that if the graph $\Sigma(\Omega)$ is $k$-magic then $k$ should be an even number. (Figure 5 illustrates the case for $\Sigma(P_2, P_2, P_3, P_4, P_5, P_5)$).

![Diagram](image)

Figure 5.

Case 2. $\#o < \#e$.

Without lost of generality, we assume the $\#o$ paths of odd length are arranged first in $\Omega$ and the remaining are $\#e$ paths of even length.
Then we label the edges of $\Sigma(\Omega)$ corresponding to the edges of the $2m$ paths $P$ starting from the edge incident with $u$ by a and $-a$ alternately from top to bottom.

Next, we label the edges of $\Sigma(\Omega)$ corresponding to the edges of the path $P_n$ starting from the edge incident with $u$ by a and $-a$ alternately or $-a$ and $a$ alternately. If $n$ is even, the last edge incident on $v$ will have the same edge labels as the first edge incident on $u$. If $n$ is odd, the last edge incident on $v$ will have the inverse edge labels of the label of the first edge incident on $u$.

Then we observe the vertex $v$ of $\Sigma(\Omega)$ will have labels $-2a \equiv 0 \pmod{k}$. This shows that if the graph $\Sigma(\Omega)$ is $k$-magic then $k$ should be an even number. (Figure 6 illustrates the case for $\Sigma(P_2, P_2, P_2, P_2, P_3, P_4, P_5, P_5)$)

![Figure 6](image)

Hence, $\text{IM}(\Sigma(\Omega)) = 2N$. $\Box$

(b) |$|l|$ is odd.

Assume |$|l|$ = 3. Now, let $P_m$, $P_n$, and $P_t$ be three paths, where $m$, $n$, and $t$ are at least 2 and $m$, $n$, and $t$ are the number of vertices of each of the paths $P_m$, $P_n$, and $P_t$ correspondingly. We denote $\Sigma(\Omega)$ by $\Theta(P_m, P_n, P_t)$.

**Theorem 2.3.** $\text{IM}(\Theta(P_m, P_n, P_t)) = N - \{2\}$, where $m$, $n$, and $t$ are even or $m$, $n$, $t$ are odd and greater or equal 3.

**Proof.** Since the graph $\Theta(P_m, P_n, P_t)$ has a vertex of degree 2 and a vertex of degree 3, therefore it is not 2-magic. Now we want to show that it is $k$-magic for any $k \neq 2$.

**Case 1.** Assume $m$, $n$, and $t$ are even and greater than 3.

We want to show that it admits a $k$-magic labeling with sum 0. Let $u$ be a vertex of degree 3 of $\Theta(P_m, P_n, P_t)$. We label the edges of $\Theta(P_m, P_n, P_t)$ corresponding to the edges of the path $P_m$ starting from the edge incident with $u$ by $a$ and $-a$ alternately. Next, we label the edges of $\Theta(P_m, P_n, P_t)$ corresponding to the edges of the path $P_n$ starting from the edge incident with $u$ by $b$ and $-b$ alternately.
We observe the edges of $P_m$, $P_n$, and $P_t$ which connect to the end vertex $v$ of degree 3 will have labels $a$, $b$, $-(a+b)$. Thus $v$ has vertex label 0. This shows that $\theta(P_m, P_n, P_t)$ is $k$-magic.

**Case 2.** Assume $m$, $n$, and $t$ are odd and greater than 2.

We want to show that it admits a $k$-magic labeling with sum 0. Let $u$ be a vertex of degree 3 of $\theta(P_m, P_n, P_t)$. We label the edges of $\theta(P_m, P_n, P_t)$ corresponding to the edges of the path $P_m$ starting from the edge incident with $u$ by $a$ and $-a$ alternately. Next, we label the edges of $\theta(P_m, P_n, P_t)$ corresponding to the edges of the path $P_n$ starting from the edge incident with $u$ by $b$ and $-b$ alternately.

We observe the edges of $P_m$, $P_n$, and $P_t$ which connect to the end vertex $v$ of degree 3 will have labels $-a$, $-b$, $(a+b)$. Thus $v$ has vertex label 0. This shows that $\theta(P_m, P_n, P_t)$ is $k$-magic. $\square$

**Example 3.** Figure 7 illustrates the result for $\theta(P_3, P_5, P_5)$ and $\theta(P_2, P_4, P_6)$

![Figure 7](image)

**Theorem 2.** $IM(\theta(P_m, P_n, P_t)) = 2N - \{2\}$, where exactly one of the numbers $m$, $n$, and $t$ is odd or where $m$, $n$, and $t$ are greater than 2 and exactly two of the numbers $m$, $n$, and $t$ are odd.

**Proof.** Since the graph $\theta(P_m, P_n, P_t)$ has a vertex of degree 2 and a vertex of degree 3, therefore it is not 2-magic. Now we want to show that it is $k$-magic for any $k \neq 2$.

**Case 1.** Assume one of the numbers $m$, $n$, and $t$ is odd. Without loss of generality let $t$ be odd. Let $u$ be a vertex of degree 3 of $\theta(P_m, P_n, P_t)$. Consider an $k$-magic labeling of $\theta(P_m, P_n, P_t)$. Then we label the edges of $\theta(P_m, P_n, P_t)$ corresponding to the edges of the path $P_m$ starting from the edge incident with $u$ by $a$ and $-a$ alternately. Next, we label the edges of $\theta(P_m, P_n, P_t)$ corresponding to the edges of the path $P_n$ starting from the edge incident with $v$ by $b$ and $-b$ alternately. Finally, we label the edges of $\theta(P_m, P_n, P_t)$ corresponding to the
edges of the path $P_t$ starting from the edge incident with $u$ by $-(a + b)$ and $a + b$, alternately.

Then we observe the edges of $P_m$, $P_n$ and $P_t$, which connect to the end vertex $v$ of degree 3 will have labels $a$, $b$, $(a + b)$. Thus $v$ has vertex label $2(a + b) ≡ 0 \pmod{k}$. This shows that if it is $k$-magic then $k$ should be an even number except 2.

Case 2. Let $m$, $n$, and $t$ are greater than 2 and exactly two of the numbers $m$, $n$, and $t$ are odd. Without loss of generality we let $n$ and $t$, be odd. Let $u$ be a vertex of degree 3 of $\theta(P_m, P_n, P_t)$. Consider an $k$-magic labeling of $\theta(P_m, P_n, P_t)$. Then we label the edges of $\theta(P_m, P_n, P_t)$ corresponding to the edges of the path $P_m$ starting from the edge incident with $u$ by $a$ and $-a$ alternately. Next, we label the edges of $\theta(P_m, P_n, P_t)$ corresponding to the edges of the path $P_n$ starting from the edge incident with $v$ by $b$ and $-b$ alternately. Finally, we label the edges of $\theta(P_m, P_n, P_t)$ corresponding to the edges of the path $P_t$ starting from the edge incident with $u$ by $-(a + b)$ and $a + b$ alternately.

Then we observe the edges of $P_m$, $P_n$, and $P_t$ which connect to the end vertex $v$ of degree 3 will have labels $a$, $-b$, $(a + b)$. Thus $v$ has vertex label $2a ≡ 0 \pmod{k}$. This shows that if it is $k$-magic then $k$ should be an even number except 2.

Hence, $\text{IM}(\theta(P_m, P_n, P_t)) = 2N - \{2\}$. □

**Example 4.** Figure 8 illustrates the result for $\theta(P_2, P_4, P_5)$ and $\theta(P_4, P_5, P_5)$.

![Diagram](image)

**Figure 8.**

We have the following surprising result.

**Theorem 2.5.** Given $\Omega = \{(P_i, \{u_i, v_i\}) : i \in I\}$ with $|I|$ odd greater than 3, we have the integer magic spectrum

(a) $\text{IM}(\Sigma(\Omega)) = N \setminus \{2\}$, if $\#o$ is even

(b) $\text{IM}(\Sigma(\Omega)) = 2N \setminus \{2\}$, if $\#o$ is odd

**Proof.** It is clear that $\Sigma(\Omega)$ is not 2-magic.

(a) If $\#o$ is even then $\#e$ is odd.
For any $k \in \mathbb{N}\backslash\{2\}$. We want to show that $\Sigma(\Omega)$ is k-magic.

Let $u$ be a vertex with maximum degree in $\Sigma(\Omega)$. Consider the induced subgraph $G$ of $\Sigma(\Omega)$ which consists of all $\#_o$ paths of odd length and $\#_e$-paths with even length. It is obvious that it has a k-magic labeling with zero sum.

Now apply Theorem 2.3. for the remaining unlabeled subgraph of $\Sigma(\Omega)$, which is of the form $\theta(P_m, P_n, P_3)$ with three odd path length. We can have a k-magic labeling $\theta(P_m, P_n, P_3)$ with zero sum. Combine this labeling with the one in $G$, we can have a k-magic labeling of $\Sigma(\Omega)$.

(b) If $\#_o$ is odd then $\#_e$ is even.

For any $k \in 2\mathbb{N}\backslash\{2\}$. We want to show that $\Sigma(\Omega)$ is k-magic.

Let $u$ be a vertex with maximum degree in $\Sigma(\Omega)$. Consider the induced subgraph $G$ of $\Sigma(\Omega)$ which consists of all $\#_o$-1 paths of odd length and $\#_e$-2 path with even lengths. It is obvious that it has a k-magic labeling with zero sum. Now apply Theorem 2.4. for the remaining unlabeled subgraph of $\Sigma(\Omega)$, which is of the form $\theta(P_m, P_n, P_3)$ with two odd path length and one even path length. We can have a k-magic labeling $\theta(P_m, P_n, P_3)$ with zero sum. Combine this labeling with the one in $G$, we have a k-magic labeling of $\Sigma(\Omega)$.

We illustrate the above result by the following:

**Example 5.** We illustrate the above results for $\text{IM}(\theta(P_2, P_4, P_4, P_5, P_5))$ and $\text{IM}(\theta(P_2, P_3, P_4, P_5, P_5))$.

\[ \theta(P_2, P_4, P_4, P_5, P_5) \quad \#_o = 2 \]

\[ \theta(P_2, P_3, P_4, P_5, P_5) \quad \#_o = 3 \]

**Figure 9.**
3. General cases.

Let \( \Omega(n, t, s) = \{(P_i, \{u_i, v_i\}) : P_i = P_n \text{ for each } i = 1, 2, \ldots, s \text{ and } d(u_i, v_i) = t \} \) is a collection of paths with two distinguish vertices \( u_i \) and \( v_i \) with distance \( t \).

Then \( \Sigma(\Omega(n, n-1, s)) \) is the generalized theta graph.

**Theorem 3.1** IM(\( \Sigma(\Omega(n, n-r, s)) \)) = \( \emptyset \) if \( r > 3 \).

**Proof.** If \( r > 2 \), the graph \( \Sigma(\Omega(n, n-r, s)) \) has a pendant path of length 2. Hence it is not \( k \)-magic for any \( k \). \( \square \)

**Theorem 3.2** If \( n \) is an odd integer greater than 3, then the integer magic spectrum \( \text{IM}(\Sigma(\Omega(n, n-3, s))) = (\cup p_i^{a_i} N) \setminus \{2\} \) where \( 3s-2 = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m} \).

**Proof.** We note first \( \Sigma(\Omega(n, n-3, s)) \) cannot be 2-magic.

Let \( u \) and \( v \) be two non-end vertices of \( P_n \) whose distance is \( n-3 \). Suppose we have \( s \) copies of \( P_n \) and we form the two vertex sum \( \Sigma(\Omega(n, n-3, s)) \) of \( \Omega(n, n-3, s) \).

Suppose we label the edges in the first path \( P_n \) of \( \Sigma(\Omega(n, n-3, s)) \) by \( a, x_1, a-x_1, x_1, a-x_1, \ldots, a-x_1, a \).

We label the edges in the second path \( P_n \) of \( \Sigma(\Omega(n, n-3, s)) \) by \( a, x_2, a-x_2, x_2, a-x_2, \ldots, a-x_2, a \).

For the \( i \)th path \( P_m \), where \( i < s \), we label the edges in the \( i \)th path \( P_n \) of \( \Sigma(\Omega(n, n-3, s)) \) by \( a, x_i, a-x_i, x_i, a-x_i, \ldots, a-x_i, a \).

Finally, for the \( s \)th path \( P_n \) of \( \Sigma(\Omega(n, n-3, s)) \) we will label the edges by \( a, -(s-1)a-(x_1 + x_2 + \ldots + x_{s-1}), sa + (x_1 + x_2 + \ldots + x_{s-1}), -(s-1)a-(x_1 + x_2 + \ldots + x_{s-1}), \ldots, sa + (x_1 + x_2 + \ldots + x_{s-1}), a \).

In order the graph \( \Sigma(\Omega(n, n-3, s)) \) is \( k \)-magic for some \( k \), we must have the vertex label for the vertex \( v \) in \( \Sigma(\Omega(n, n-3, s)) \) is \( a \). That is the sum of the edge labels of edges incident on \( v \) is 
\[(a-x_1) + (a-x_2) + \ldots + (a-x_{s-1}) + sa + (x_1 + x_2 + \ldots + x_{s-1}) + sa = a \]
implicating that \( (3s-2) a = 0 \).
Thus if \( 3s-2 = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m} \), then \( \text{IM}(\Sigma(\Omega(n,n-3,s))) = (\cup p_i^{a_i} N) \setminus \{2\} \). \( \square \)

**Example 6.** We illustrate the above results for \( \text{IM}(\Sigma(\Omega(7, 4, 2))) \) and \( \text{IM}(\Sigma(\Omega(5, 2, 3))) \)

![Diagram](image)

\( \text{IM}(\Sigma(\Omega(7, 4, 2))) = 4N \)

\( \text{IM}(\Sigma(\Omega(5, 2, 3))) = 7N \)

**Figure 10.**
Example 7. The integer magic spectrum for $\text{IM}(\Sigma(\Omega(5,2,4)))$ is $2N \cup 5N \setminus \{2\}$.

![Graph 1](image1)

$\Sigma(\Omega(5,2,4))$ is 4-magic

![Graph 2](image2)

$\Sigma(\Omega(5,2,4))$ is 5-magic

Figure 11.

Similarly, we have the following results.

**Theorem 3.3** If $n$ is even integer greater than 3, then we have the integer magic spectrum $\text{IM}(\Sigma(\Omega(n,n-3,s))) = N \setminus \{2\}$ for all $s \geq 2$.

Since the proof is similar to Theorem 3.2, hence we skip the proof.

Example 8. Figure 12 illustrates the above result.

![Graph 3](image3)

$\text{IM}(\Sigma(\Omega(6,3,2))) = N \setminus \{2\}$

Figure 12.

**Theorem 3.4**. If $n$ is odd integer greater than 3, then the integer magic spectrum $\text{IM}(\Sigma(\Omega(n,n-2,s))) = \cup p_i^{a_i}N$ where $s = p_1^{a_1}p_2^{a_2} \ldots p_m^{a_m}$

Example 9. We show the above result by two examples.

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Theorem 3.5. If $n$ is even integer greater than 3, then the integer magic spectrum $\text{IM}(\Sigma(\Omega(n,n-2,s))) = (\cup \, p_i \, a_i) \setminus \{2\}$ where $2s-2 = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$

Example 10. The following two examples illustrates the above result.

References


[17] S-M Lee, E. Salehi and Hugo Sun, Integer-magic spectra of trees with diameters at most four, manuscript.

[19] S-M Lee, L. Valdes and Yong-Seng Ho, On group-magic index sets of
double trees, abbreviated trees and trees, *The Journal of Combinatorial

[20] S-M Lee, and Henry Wong, On the integer-magic spectra or the power of

in JCMCC 2003.

[22] L. Sandorova and M. Trenkler, On a generalization of magic graphs, in


graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing*

[26] R.P. Stanley, Linear homogeneous diophantine equations and magic

[27] R.P. Stanley, Magic labeling of graphs, symmetric magic squares, systems

1031-1059.

[29] B.M. Stewart, Supermagic complete graphs, *Canadian Journal of


[31] M. Trenkler, Some results on magic graphs, in "Graphs and other
Combinatorial Topics", Proc. Of the 3rd Czechoslovak Symp., Prague, 1983,
edited by M. Fieldler, Teubner-Texte zur Mathematik Band, 59(1983), Leipzig,
328-332.